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ALTERNATIVES IN CALCULATING DOUBLE INTEGRALS

***Riad Zaidan**

Department of Mathematics, College of Science and Information Technology, Al-Quds Open University, Jenin, Palestine

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ABSTRACT

In this paper, alternatives in calculating double integrals will be used instead of the direct difficult known methods.

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INTRODUCTION

In order to integrate a function with more than one variable((here two variables)) , we integrate first with respect to one variable and treat the other one as a constant, similar to the process of partial differentiation of functions of several variables.

We first look at some examples in double integrals of the form

$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ in which the region R of integration is defined as

$$R = \{(x,y): a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

or

$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$ in which the region R of integration is defined as

$$R = \{(x,y): c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

Note that the internal variable of integration may be a function of the external one, but the external one must have constant limits.

Example 1: Evaluate the double integral $\int_2^4 \int_0^1 (2x+y)\,dy\,dx$.

****Corresponding author:*** **Riad Zaidan,**

Solution:

$$\int_2^4 \int_0^1 (2x + y)\, dy\, dx = \int_2^4 (2xy + \frac{y^2}{2}\Big]_0^1)\, dx$$

$$= \int_2^4 ((2x(1) + \frac{1^2}{2}) - 0)dx = \int_2^4 (2x + \frac{1}{2})dx = (x^2 + \frac{1}{2}x\Big]_2^4$$

$$= 4^2 + \frac{1}{2}(4) - (2^2 + \frac{1}{2}(2)) = 13.$$

Example 2: Evaluate the double integral $\iint_R \frac{y}{1+x^2} dA$

where $R = \{(x, y) \mid 0 \le x \le 4 \ and \ 0 \le y \le \sqrt{x}\}$.

Solution:

$$\iint_R \frac{y}{1+x^2} dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx = \int_0^4 \frac{1}{1+x^2} \quad \frac{y^2}{2}\Big]_0^{\sqrt{x}} dx$$

$$= \int_0^4 (\frac{1}{1+x^2}(\frac{(\sqrt{x})^2}{2}) - 0)dx = \frac{1}{2}\int_0^4 \frac{x}{1+x^2}dx = \frac{1}{4}\ln(1+x^2)\Big]_2^4$$

$$= \frac{1}{4}(\ln 17 - \ln 5) = \frac{1}{4}\ln(\frac{17}{5})$$

Now we have some definitions.

Definition: The area of a closed, bounded plane region R is

$$\iint_R dA \qquad (1)$$

That is, the double integral when the integrated function is $f(x, y) = 1$.

Example 3: find $\iint_R dA \ : R = \{(x, y) : x^2 + y^2 \le 25\}$.

Solution: $\iint_R dA = Area \ of \ circle \ with \ radius \ r = 5 \ and \ centered \ at \ the \ origion$

So $\iint_R dA = \pi(5)^2 = 25\pi$.

If f is the area density of a thin plate covering a region R , then the double integral of f over R gives the mass of this plate ………………………(1) i.e

M = mass of R = $\iint_R f(x, y)\, dA$.

Similarly, the moment with respect to the x-axis can be calculated as

$$M_x = \iint_R y\, f(x,y)\, dA$$, and

the moment with respect to the y-axis can be calculated as

$$M_y = \iint_R x\, f(x,y)\, dA$$

Then we calculate the center of mass of R via

Center of mass of

$$R = (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) \tag{2}$$

So $\bar{x} = \dfrac{M_y}{M} = \dfrac{\iint_R x\, f(x,y)\, dA}{\iint_R f(x,y)\, dA} \Rightarrow \iint_R x\, f(x,y)\, dA = \bar{x} \,.\left(\iint_R f(x,y)\, dA\right)$

And $\bar{y} = \dfrac{M_x}{M} = \dfrac{\iint_R y\, f(x,y)\, dA}{\iint_R f(x,y)\, dA} \Rightarrow \iint_R y\, f(x,y)\, dA = \bar{y} \,.\left(\iint_R f(x,y)\, dA\right)$

Now, a special case of integrals will be calculated in two ways, the traditional approach , and a new approach related to the center of mass.

Example 4: find $\iint_R y\, dA \quad : R = \left\{ (x,y) : x^2 + (y-3)^2 \le 100 \right\}$

Solution: We first use the traditional approach to find this integral, and after that the new method is used for the same purpose. The region R is the circular desk centered at $(0,3)$ with radius $r = 10$, so the region R can be determined as

$$R = \left\{ (x,y) :, -\sqrt{100-(y-3)^2} \le x \le \sqrt{100-(y-3)^2}, -7 \le y \le 13 \right\}$$

So,

$$\iint_R y\, dA = \iint_R y\, dx\, dy = \int_{-7}^{13} \int_{-\sqrt{100-(y-3)^2}}^{\sqrt{100-(y-3)^2}} y\, dx\, dy =$$

$$\int_{-7}^{13} y\, x \Big]_{-\sqrt{100-(y-3)^2}}^{\sqrt{100-(y-3)^2}} dy = \int_{-7}^{13} y\left(\sqrt{100-(y-3)^2} - (-\sqrt{100-(y-3)^2}\right) dy$$

$= \int_{-7}^{13} y\left(2\sqrt{100-(y-3)^2}\right) dy$, then we use integration by substitution as follows:

Let $y - 3 = 10 \cos\theta \Rightarrow dy = -10 \sin\theta \, d\theta$, changing the limits of integration we have

$$\begin{aligned} y = -7 &\Rightarrow -7-3 = 10\cos\theta \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi \\ y = 13 &\Rightarrow 13-3 = 10\cos\theta \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0 \end{aligned}$$ so

$$= \int_{\pi}^{2\pi} 2(3+10\cos\theta)(\sqrt{100-100\cos^2\theta}\ .\ (-10\sin\theta)d\theta$$

$$= \int_{\pi}^{2\pi} 2(3+10\cos\theta)(\sqrt{100(1-\cos^2\theta)}\ \ .\ (-10\sin\theta)d\theta$$

$$= \int_{\pi}^{2\pi} 2(3+10\cos\theta)(\sqrt{100\sin^2\theta}\ \ .\ (-10\sin\theta)d\theta$$

$$= \int_{\pi}^{2\pi} 2(3+10\cos\theta)10\left|\sin\theta\right|.\ (-10\sin\theta)d\theta \quad but\ \pi \le \theta \le 2\pi \Rightarrow \left|\sin\theta\right| = -\sin\theta$$

Therefore

$$= \int_{\pi}^{2\pi} 2(3+10\cos\theta)10\left|\sin\theta\right|.\ (-10\sin\theta)d\theta$$

$$= \int_{\pi}^{2\pi} 2(3+10\cos\theta)10(-\sin\theta).\ (-10\sin\theta)d\theta$$

$$= 200\int_{\pi}^{2\pi} (3+10\cos\theta)\sin\theta\ .\ \sin\theta\, d\theta$$

$$= 200\int_{\pi}^{2\pi} (3+10\cos\theta)\sin^2\theta\ d\theta$$

$$= 200\int_{\pi}^{2\pi} (3\sin^2\theta + 10\sin^2\theta\cos\theta)\, d\theta \ \ \ldots\ldots\ldots\ldots..(*)$$

By using the identity $\sin^2\theta = \frac{1}{2} - \frac{\cos(2\theta)}{2}$ in the first term and integral by substitution in the second term (*) becomes:

$$= 200\int_{\pi}^{2\pi} (3\sin^2\theta + 10\sin^2\theta\cos\theta)\, d\theta \ = 200\int_{\pi}^{2\pi} (3(\frac{1}{2} - \frac{\cos(2\theta)}{2}) + 10\sin^2\theta\cos\theta)\, d\theta$$

$$= 200\left[\left. 3(\frac{1}{2}\theta - \frac{\sin(2\theta)}{4}) + 10\frac{\sin^3\theta}{3})\right]_{\pi}^{2\pi}\ \right]$$

$$= 200\left[3(\frac{1}{2}(2\pi) - \frac{\sin(4\pi)}{4}) + 10\frac{\sin^3(2\pi)}{3}) - (3(\frac{1}{2}(\pi - \frac{\sin(\pi)}{4}) + 10\frac{\sin^3(\pi)}{3})\right]$$

$$= 200\left[3\pi - 0) + 0) - \frac{3\pi}{2} - 0 - 0\right]$$

$$= 200(\frac{3\pi}{2}) = 300\pi \ldots\ldots\ldots\ldots. Q.E.D$$

We noticed the heavy work and difficulty in calculating the previous integral because of using the direct calculation for that integral. Next we will calculate it by means of the center of mass as follows:

$$\bar{y} = \frac{\iint_R y\ f(x,y)\, dA}{\iint_R f(x,y)\, dA} \Rightarrow \iint_R y\ f(x,y)\, dA = \bar{y}(\iint_R f(x,y)\, dA)$$

In our example, we notice that $f(x, y) = 1$ so the center of mass is the same as the center of the circle

$x^2 + (y-3)^2 = 100$ i.e the point $(\bar{x}, \bar{y}) = (0,3)$, so $\bar{y} = 3$.

Therefore

$\iint_R y\ dA = \bar{y}\ .\iint_R dA$ but $\iint_R dA =$ area of the desk with radius $r = 10$ so

$\iint_R dA = \pi(10)^2 = 100\pi$ and so we have

$$\iint_R y\ dA = (3).(100\pi) = 300\pi\ \cdot$$

Example 5: find $\iint_R x\ dA\ \ : R = \left\{(x,y): 25(x-5)^2 + 16(y+2)^2 \le 400\right\}$.

Solution: We first look at the nature of the region R :

$25(x-5)^2 + 16(y+2)^2 \le 400 \Rightarrow \dfrac{25(x-5)^2}{400} + \dfrac{16(y+2)^2}{400} \le 1$ so

$$\frac{(x-5)^2}{16} + \frac{(y+2)^2}{25} \le 1$$

which is the interior together with the boundary of an ellipse centered at $(5,-2)$, with major axes length $= 2a = 2(5) = 10$ and minor axes length $= 2b = 2(4) = 8$. So as we know, the area of this ellipse is $\pi a b$, and in this case $A = \pi a b = \pi(5)(4) = 20\pi$. Now, we go back to our problem.

We note that

$$\bar{x} = \frac{\iint_R x\ f(x,y)\,dA}{\iint_R f(x,y)\,dA} \Rightarrow \iint_R x\ f(x,y)\,dA = \bar{x}\,.(\iint_R f(x,y)\,dA),$$

but when $f(x,y) = 1$ we have $\iint_R x\ dA = \bar{x}\,.\iint_R dA$.

Because of symmetry on an ellipse, we have the point $(5,-2)$ as the center of mass so $\bar{x} = 5$. Now, back to our integral we get

$\iint_R x\ dA = \bar{x}\,.\iint_R dA$ =5(area of the ellipse)= $5\ A = 5\pi(5)(4) = 100\pi$.

By looking at the two previous methods in example 4, we can notice the difference between the times that each one took , so it is preferable to use the second method, as we did in example 5. Finally, many of integrals can be treated as the previous one, in which time will be minimized as much as possible.

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