Available online at http://www.journalijdr.com

IJDR

ISSN: 2230-9926

International Journal of Development Research
Vol. 13, Issue, 02, pp. 61755-61765, February, 2023
https://doi.org/10.37118/ijdr.26338.02.2023

RESEARCH ARTICLE OPEN ACCESS

INTRINSIC METRICS AND GEOMETRY OF DIRICHLET FORMS

***Hanana Elnage**

Department of Mathematics, King Khalid University & Bakht Elruda University, College of sciences , S.U.D, S.U.A

ARTICLE INFO

Article History:
Received 11th January, 2023
Received in revised form
17th January, 2023
Accepted 08th February, 2023
Published online 28th February, 2023

KeyWords:

Dirichlet forms

**Corresponding author: Hanana Elnage*

ABSTRACT

We present a general conception of intrinsic metric and study some of its properties. We provide for general regular Dirichlet forms. Given a regular, strongly local Dirichlet form Ɛ, the local doubling and local Poincaré inequalities are satisfied, we obtain that: the intrinsic differential and distance structures of Ɛ coincide.

Citation: ***Hanana Elnage. 2023.*** "Intrinsic metrics and geometry of dirichlet forms*", International Journal of Development Research*, 13, (02), 61755-61765.

1. INTRODUCTION

The aim of this paper is to propose an extension of the conception of intrinsic metric from strongly local Dirichletforms to the general case, and show that if the measure space, equipped with the intrinsic metric associated with a strongly local Dirichlet form, is doubling, supports a (1, 2)-Poincaré inequality. In Section 2 we then present a general concept of intrinsic metric. In Section 3, we recall some basic property of general regular strongly local Dirichlet forms Ɛ, and present the weak coincidence of the intrinsic distance and differential structures of $\mathcal{J}$ established in [8]. Some finer properties, which are essentially established in [7,9], are also given with the additional local Poincaré and doubling assumptions; see Lemma 3.4 and Lemma 3.5.

Let X be a locally compact, separable metric space, m a positive Radon measure on C with $\text{supp} m = X$. The functions on C we consider will all be real valued. By $\omega_0(X)$ denote the set of continuous function on X with compact support and Ɛ is regular Dirichlet form. The$\wp$ is a pseudo-metric on $X$and $A^2 \subset X$, a pseudo-metric $\wp: X \times X \to [0, \infty]$ is called an intrinsic metric with respect to the Dirichlet form Ɛ, A map $\wp: X \times X \to [0, \infty]$ if $\wp(x_n, x_n) = 0, \wp(x_n, x_{n+1}) = \wp(x_{n+1}, x_n)$, and $\wp(x_n, x_{n+1}) \leq \wp(x_n, x_{n+2}) + \wp(x_{n+2}, x_{n+1})$ for all $x_n, x_{n+1}, x_{n+2} \in C$. That

$$\sum \left|\wp(x_n, x_{n+1}) - \wp\left(x_n', x_{n+1}'\right)\right| \leq \wp \sum (x_n, x_n') + (x_{n+1}, x_{n+1}').$$

We emphasize that $\wp$ may not be continuous with respect to the original topology.

As $\wp$is a pseudo-metric, then so is $\wp \wedge \mathcal{T}$for any $\mathcal{T} \geq 0$. That

$$(\wp \wedge \mathcal{T})_{A^2} = \wp_{A^2} \wedge \mathcal{T},$$

and the estimate

$$\sum |\wp_{A^2}(x_n) \wedge \mathcal{T} - \wp_{A^2}(x_{n+1}) \wedge \mathcal{T}| \leq \wp \sum (x_n, x_{n+1}).$$

We assumption tow spaces of all measurable real valued functions, one of these is $L^2(X)$ and another is $L^\infty(X, m)$, $\mathcal{M}$ is dense subspace that $\mathcal{M} \subset L^2(X, m)$.The space $\mathcal{M}_{loc}^*$ of functions locally in domain is defined to be the set of all functions $u_j \in \mathcal{M}_{loc}$.

Also $\mathcal{M}_{loc}^{*}$and the measures $\mu^{(*)}$ are well compatible with approximation via cut-off procedures see Lemma 2.5.

We can extend $\mu^{(b)}$ to the space $\mathcal{M}_{loc}^{*}$, to do that we define for $\mathcal{F} \subset X$measurable and $u_j \in \mathcal{M}_{loc}^{*}$,

$\mu^{(b)}(u_j)(\mathcal{F}) := \mu^{(b)}(u_j) := \int_{\mathcal{F}\times X-d} \left(\widetilde{u_j}(x_n) - \widetilde{u_j}(x_{n+1})\right)^2 \mathcal{L}(dx_n, dx_{n+1})$.

Proposition 1.1. For $u_j \in \mathcal{M}_{loc}^{*}$, the map $\mu^{(b)}(u_j)(\cdot)$ is a Radon measure.

Proof: We only have to show, that $\mu^{(b)}$ is inner regular, the rest is obvious. For this let $\mathcal{F} \subset X$be measurable. As $\mathcal{L}$is a Radon measure, $\mu^{(b)}(u_j)(\mathcal{F})$ can be approximated by $\int_{\varphi} \left(\widetilde{u_j}(x_n) - \widetilde{u_j}(x_{n+1})\right)^2 \mathcal{L}(dx_n, dx_{n+1})$ with $\varphi \subset \mathcal{F} \times X - d$compact. But then $\mu^{(b)}(u_j)(\varphi') = \int_{\varphi'\times C-d} \left(\widetilde{u_j}(x_n) - \widetilde{u_j}(x_{n+1})\right)^2 \mathcal{L}(dx_n, dx_{n+1})$. with $\varphi' := \{x_n \in X : \exists x_{n+1} \in C\text{with}(x_n, x_{n+1}) \in \varphi\}$with the projection from φon the first component also approximates $\mu^{(b)}(u_j)(\mathcal{F})$. As $\varphi'$is compact the desired regularity follows.

We proved that the intrinsic distance $d_{\mathcal{E}}$ of Dirichlet form $\mathcal{E}$ is given by the original distance d on X, and hence that the intrinsic differential and distance structures of $\mathcal{E}$ coincide, that is, for all $u_j \in \mathrm{Lip}_{d_J}(X)$

$$\frac{d}{dp}\Gamma(u_j, u_j) = \left(\mathrm{Lip}_{d_J} u_j(X)\right)^2, \quad a.e \qquad (1)$$

Relies on a weak coincidence (much weaker than (1) of the intrinsic distance and differential structures given by Lemma 3.1, which holds for general regular strongly local Dirichlet forms.

1. *Intrinsic metrics and their applications*

Accessing strongly local Dirichlet forms the intrinsic metric is a punchy tool. It has been used in studying decay of heat kernels, the investigation of Harnack inequalities and to get good cut-off functions in the study of spectral properties cf [2, 3, 4, 12, 13]. First we introduced some define.

Definition 2.1. Let $\mathcal{R}_b$and $\mathcal{R}_{b+\epsilon}$ is two Radon measures, that $\mathcal{R}_b + \mathcal{R}_{b+\epsilon} \leq \mathcal{R}$such that for all $A^2 \subset X$and all $\mathcal{T} > 0$,

$\wp_{A^2} \wedge \mathcal{T} \in \mathcal{M}_{loc}^{*} \cap \omega(X)$

also

$\mu^{(b)}(\wp_{A^2} \wedge \mathcal{T}) \leq \mathcal{R}_b \quad and \ \mu^{(b+\epsilon)}(\wp_{A^2} \wedge \mathcal{T}) \leq \mathcal{R}_{b+\epsilon}$, the standard Euclidean distance, $\wp(x_n, x_{n+1}) := |x_n - x_{n+1}|$, is an intrinsic metric for $\mathcal{E}$. We have norm definition follow.

Definition 2.2. Let $\mathcal{F} \subset X$and $s \geq 0$. Hence, $\mathcal{F}$ is linked to cut-off function into extent s,

$\xi_{\mathcal{F},s}(x_n) := (1 - \wp\mathcal{F}(x_n)/s)^+$,

for some ball $B := B(x_n, d)$, where d is radius the intrinsic impart $B_d(\mathcal{F}) := \{x_n \in X: \wp\mathcal{F}(x_n) \leq d\}$, when $\mathcal{F}$ is a set, the intrinsic boundary its $A_d^2(\mathcal{F}) := B_s(\mathcal{F}) \cap B_s(\mathcal{F}')$.

Now we show some properties of intrinsic metrics

Proposition 2.3. let $A^2 \subset X$,and $\wp_{(A^2)}(x_n) < \infty$for all $x_n \in X$, let $\wp$ be an intrinsic. Then $\wp_{(A^2)} \in \mathcal{M}_{loc}^{*} \cap \omega(X)$, $\epsilon \geq 0$ and $\mu^{(b+2\epsilon)}(\wp_{(A^2)}) \leq \mathcal{R}$.

Proof: See Definition 2.1,$\wp_{A^2} \wedge \mathcal{T}$for any $\mathcal{T}$. We have $(\wp_{A^2} \wedge \mathcal{T}) \in \mathcal{M}_{loc}^{*}$ and $\mu^{(b+2\epsilon)}(\wp_{A^2} \wedge \mathcal{T}) \leq \mathcal{R}_b + \mathcal{R}_{(b+\epsilon)} = \mathcal{R}$, for any $\mathcal{T} > 0$.

Proposition 2.4. Let ω be a bound and $\wp$ is an intrinsic metric, $\mathcal{F} \subset X$, $s > 0$. Then $\xi_{\mathcal{F},s} \in \mathcal{M}_{loc}^{*} \cap \omega(X)$ and $\mu^{(b+2\epsilon)}(\xi_{\mathcal{F},s}) \leq (1/s^2)\mathcal{R}$. Moreover, if $B_s(\mathcal{F})$ is relatively compact, then $\xi_{\mathcal{F},s} \in \mathcal{M} \cap \omega_0(X)$.

Proof: $\mathcal{G}\,\mathcal{F}$is continuous, $\xi_{\mathcal{F},s}$is so as well. Moreover, $\mathcal{G}\,\mathcal{F}$belongs to $\mathcal{M}_{loc}^{*}$by Proposition 2.3and as Dirichlet form, $\mathcal{F}$is compatible with cut-off procedures. Hence $\xi_{\mathcal{F},s} \in \mathcal{M}_{loc}^{*}$. In order to show the claimed upper bound on $\mu^{(b+2\epsilon)}(\xi_{\mathcal{F},s})$, we recallthat $\mu^{(b+\epsilon)}(\xi_{\mathcal{F},s}) \leq (1/s^2)\mu^{(b+\epsilon)}(\mathcal{G}\mathcal{F})$, see [5]. Moreover, since $|\xi_{\mathcal{F},s}(x_n) - \xi_{\mathcal{F},s}(x_{n+1})| \leq (1/s)|\mathcal{G}\mathcal{F}(x_n) - \mathcal{G}\mathcal{F}(x_{n+1})|$,we have $\mu^{(b)}(\xi_{\mathcal{F},s}) \leq (1/s^2)\mu^{(b)}(\mathcal{G}\mathcal{F})$. Therefore, thebound $\mu^{(b+2\epsilon)}(\wp\mathcal{F}) \leq \mathcal{R}$from Proposition 2.3implies the bound $\mu^{(b+2\epsilon)}(\xi_{\mathcal{F},s}) \leq (1/s^2)\mathcal{R}$.

Here need introduce result about intrinsic metrics in follow Lemmas.

Lemma 2.5. Let $u_j \in L_{loc}^{\infty} \cup \mathcal{M}_{loc}$and assume that there is a Radon measure $\mathcal{R}_1$ such that for every $\mathcal{T} > 0$ one has $(u_j)_{\mathcal{T}} := (u_j \wedge \mathcal{T}) \vee (-\mathcal{T}) \in \mathcal{M}_{loc}^{*}$and $\mu^{(b)}(u_j)_{\mathcal{T}} \leq \mathcal{R}_1$. Then $u_j \in \mathcal{M}_{loc}^{*}$and $\mu^{(*)}(u_j) = \lim_{\mathcal{T}\to\infty} \mu^{(*)}(u_j)_{\mathcal{T}}$ for $* = b, (b+\epsilon), (b+2\epsilon)$. In particular, $\mu^{(b)}(u_j) \leq \mathcal{R}_1$.

Proof: Note that $u_j \in L_{loc}^{\infty}$agrees locally with $(u_j)_{\mathcal{T}}$for $\mathcal{T}$big enough. Thus, obviously $u_j$belongs to $\mathcal{M}_{loc}$. Therefore we only have to show that

$$\int_{\varphi\times X-d} \left(\widetilde{u_j}(x_n) - \widetilde{u_j}(x_{n+1})\right)^2 \mathcal{L}\left(d(x_n), d(x_{n+1})\right) < \infty.$$

This follows as $\left(\left(u_j\right)_{\mathcal{T}}(x_n)-\left(u_j\right)_{\mathcal{T}}(x_{n+1})\right)^2$ converges monotonically to $\left(u_j(x_n)-u_j(x_{n+1})\right)^2$ and

$$\int_{\varphi\times X-d}\left(\left(u_j\right)_{\mathcal{T}}(x_n)-\left(u_j\right)_{\mathcal{T}}(x_{n+1})\right)^2\mathcal{L}\,d(x_n),d(x_{n+1})\le\mathcal{R}_1(\varphi)<\infty,$$

uniformly in $\mathcal{T}$. For $*=b,(b+\epsilon)$the convergence of $\beta^{(*)}\left(\left(u_j\right)_{\mathcal{T}}\right)$ is also clear by monotone convergence. To deal with $*=(b+2\epsilon)$, i.e., the strongly local part, we note that $\left(v_j\right)_{\mathcal{T}}\to v_j$ with respect to $\mathcal{E}_1$ for all $v_j\in\mu$, see [5].

Lemma 2.6. Let $\wp$be an intrinsic metric. Then $\int_{\mathscr{F}\times X-d}\wp^2(x_n,x_{n+1})\,\mathscr{L}(dx_n,dx_{n+1})\le\mathscr{R}_b(\mathscr{F})$, for any measurable set $\mathscr{F}\subset X$.

Proof: Let $\mathcal{E}>0$ and $r>2\mathcal{E}$be arbitrary. We first consider sets $\mathscr{F}$with $\mathscr{F}\subset B_{\mathcal{E}}(\widetilde{x_n})$ for some $\widetilde{x_n}$. Using the fact that for $x_n\in\mathscr{F}$ $\wp(x_n,x_{n+1})\le\wp(x_{n+1},\widetilde{x_n})-\wp(x_n,\widetilde{x_n})+2\wp(x_n,\widetilde{x_n})\le|\wp(x_{n+1},\widetilde{x_n})-\wp(x,\tilde{x})|+2\mathcal{E}$, we can estimate for every $\zeta>0$

$$\int_{\substack{\mathcal{F}\times X\\ \wp(x_n,x_{n+1})>r}}\wp^2(x_n,x_{n+1})\,\mathcal{L}(dx_n,dx_{n+1})$$

$$\le(1+\zeta)\int_{\mathcal{F}\times X-d}\left(\wp(x_n,\widetilde{x_n})-\wp(x_{n+1},\widetilde{x_n})\right)^2\mathcal{L}(dx_n,dx_{n+1})$$
$$+\left(1+\frac{1}{\zeta}\right)4\mathcal{E}^2\int_{\substack{\mathcal{F}\times X\\ \wp(x_n,x_{n+1})>r}}d\mathcal{L}.$$

The first term on the right side is controlled since by the definition of an intrinsic metric and by Lemma 2.5 we have

$$\int_{\mathcal{F}\times X-d}\left(\wp(x_n,\widetilde{x_n})-\wp(x_{n+1},\widetilde{x_n})\right)^2\mathcal{L}(dx_n,dx_{n+1})\le\mathcal{R}_b(\mathcal{F}).$$

In order to control the second term on the right side we estimate for $x_n\in\mathcal{F}$and $x_{n+1}\in X$with $\wp(x_n,x_{n+1})>r$ $\wp(x_{n+1},\widetilde{x_n})-\wp(x_n,\widetilde{x_n})\ge\wp(x_{n+1},x_n)-2\wp(x_n,\widetilde{x_n})\ge r-2\mathcal{E}$,

which yields $\int_{\substack{\mathcal{F}\times X\\ \wp(x_n,x_{n+1})>r}}d\mathcal{L}\le\frac{1}{(r-2\mathcal{E})^2}\int_{\mathcal{F}\times X-d}\left(\wp(x_n,\widetilde{x_n})-\wp(x_{n+1},\widetilde{x_n})\right)^2\mathcal{L}(dx_n,dx_{n+1})$.

Putting these estimates together we infer that

$$\int_{\mathcal{F}\times X-d}\wp^2(x_n,x_{n+1})\,\mathcal{L}(dx_n,dx_{n+1})\le\left(1+\zeta+\left(\frac{2\mathcal{E}}{r-2\mathcal{E}}\right)^2\left(1+\frac{1}{\zeta}\right)\right)\mathcal{R}_b(\mathcal{F}).$$

With this estimate at hand, we can now pass to arbitrary compact sets $\mathcal{F}$. An arbitrary compact $\mathcal{F}$can be covered by finitely many disjoint sets $\mathcal{F}_n$, each one being contained in an intrinsic ball $B:=B(x_n,s)$. In this way, the previous estimate extends to arbitrary compact $\mathcal{F}$. Letting first $\upsilon\to 0$, then $\zeta\to 0$ and finally $r\to 0$ we obtain the desired estimate for all compact sets $\mathcal{F}$. The general case follows from regularity.

Remarks 2.7.

(i) If $\mathcal{E}$ is a regular form on (X,p), that $\mathcal{E}\in\mathcal{M}_{loc}$, for all $u_j\in\mathcal{M}_{loc}$ the exists a quasi-continuous version $\breve{u}_j$.

(ii) If $\wp_0$ be a pseudo-metric, $\wp_1$ be an intrinsic metric, and $\wp_0\le\wp_1$.Then $\wp_0$ is an intrinsic metric

$$\sum\left|\wp_{0,A^2}(x_n)-\wp_{1,A^2}(x_{n+1})\right|\le\sum\wp_0(x_n,x_{n+1})\le\wp_1(x_n,x_{n+1}).$$

(iii) If $X=\mathbb{R}^d, d\ge 1$, with Lebesgue measure for $A^2\subset\mathbb{R}^d$and $\mathcal{T}>0$ the function $\wp_{A^2}\wedge\mathcal{T}$is Lipschitz continuous its gradient exists and equals $|\nabla(\wp_{A^2}\wedge\mathcal{T})|=1$ on $\{\wp_{A^2}<T\}$ and $\{\wp_{A^2}\ge\mathcal{T}\}$. Whenever ϱ

$$\wp_{A^2}(x_n):=\inf_{x_{n+1}\in A^2}\wp(x_n,x_{n+1}).$$

We have given sense to $\mathcal{E}(u_j,\phi)$ for $u_j\in\mathcal{M}_{loc}^*$and $\varphi\in\mathcal{M}$with compact support. In a similar way, the expression $h(u_j,\phi)$ is meaningful for $u_j\in\mathcal{M}_{loc}^*(h)$ and $\varphi\in\mathcal{M}(h)$ with compact support.

Definition 2.8. A function $u_j\in\mathcal{M}_{loc}^*(h)\backslash\{0\}$is called a generalized eigen-function corresponding to the generalized eigen value $\lambda\in\mathbb{R}$ if $h(u_j,\varphi)=\lambda(u_j,\varphi)$ for all $\varphi\in\mathcal{M}(h)$ with compact support. The next theorem gives an effective bound on the infimum of the spectrum by representing the form. It requires that the generalized eigen function has a fixed sign.

Theorem 2.9. Let $h = \mathcal{E} + v_j$ with $v_j^+ \in \mathcal{M}_0, v_j^- \in \mathcal{M}_1$ and $\mathcal{E}$ a regular Dirchlet form. Let u_j be a generalized eigen-function to the eigen-value λ with $u \neq 0$ q.e. and $u_j{}^{-1} \in \mathcal{M}_{loc}^*$. Then the formula

$$\sum h(\phi,\psi) - \lambda(\phi,\psi) = \int_{X\times X} \sum u_j(x)u_j(y)d\Gamma(\phi u_j{}^{-1}, \psi u_j{}^{-1})$$

holds true for all $\phi,\psi \in \mathcal{M}(h)$ with $\phi u_j{}^{-1}, \psi u_j{}^{-1} \in \mathcal{M}_{loc}^*(h)$ and $\phi\psi u_j{}^{-1} \in \mathcal{M}_\omega(h)$. If $u_j{}^{-1} \in \mathcal{M}_{loc}^*(h) \cap L_{loc}^\infty$ the formula holds true for all $\phi,\psi \in \mathcal{M}(h) \cap L_{loc}^\infty$.

Here, if F is a space of functions on X, F_ω denotes the subset of elements in F with compact support.

Proof: We follow the argument given in [18]. Without loss of generality we assume $\lambda = 0$ and $k = 0$. The Leibniz rule gives
$0 = \Gamma(u_j, 1) = \Gamma(u_j, u_j u_j{}^{-1}) = u_j{}^{-1}(x_n)\Gamma(u_j,u_j) + u_j(x_{n+1})\Gamma(u_j, u_j{}^{-1})$.

Using the fact that u_j is a generalized eigen function, the Leibniz rule and the preceding formula we can calculate

$$\begin{aligned}
h(\phi,\psi) &= \mathcal{E}(\phi,\psi) + v_j(\phi,\psi) = \mathcal{E}(\phi,\psi) + v_j(\phi\psi u_j{}^{-1}, u_j) = \mathcal{E}(\phi,\psi) - \int_{X\times X} u_j(x_n)u_j x^{-1} d\Gamma(\phi\psi u_j{}^{-1}, u_j)\\
&= \mathcal{E}(\phi,\psi) + \int_{X\times X} u_j(x_n)u_j(x_{n+1})d\Gamma(\phi\psi u_j{}^{-1}, u_j{}^{-1})\\
&= \mathcal{E}(\phi,\psi) + \int_{X\times X} u_j(x_n)u_j(x_{n+1})\phi(x_n)d\Gamma(\psi u_j{}^{-1}, u_j{}^{-1}) + \int_{X\times X} u_j(x_n)\psi(x_{n+1})d\Gamma(\phi, u_j{}^{-1})\\
&= \int_{X\times X} u_j(x_n)u_j(x_{n+1})\phi(x_n)d\Gamma(u_j{}^{-1}, \psi u_j{}^{-1}) + \int_{X\times X} u_j(x_n)d\Gamma(\phi, \psi u_j{}^{-1}) = \int_{X\times X} u_j(x_n)u_j(x_{n+1})d\Gamma(\phi u_j{}^{-1}, \psi u_j{}^{-1}).
\end{aligned}$$

This gives the first statement.

The argument given above can be modified to give the following results. There, we do not need the assumptions $u_j > 0$ and $u_j{}^{-1} \in \mathcal{M}_{loc}^*$ but then have stronger restrictions on ϕ and ψ.

Theorem 2.10. Let $h = \mathcal{E} + v_j$ with $v_j^+ \in \mathcal{M}_0$ and $v_j^- \in \mathcal{M}_1$. Let u_j be a generalized eigen function to the eigen value λ. Then,

$$h(u_j\phi, u_j\psi) - \lambda(u_j\phi, u_j\psi) = \int_{X\times X} u_j(x_n)u_j(x_{n+1})d\Gamma(\phi,\psi),$$

for all $\phi,\psi \in \mathcal{M}(h) \cap L_\omega^\infty$ whenever $u_j\phi, u_j\psi, u_j\phi, \psi \in \mathcal{M}(h)$. In particular, the formula holds for all $\phi,\psi \in \mathcal{M}(h) \cap L_\omega^\infty$ if $u_j \in L_\omega^\infty$.

Proof: Without loss of generality we can assume $k = 0$ and $\lambda = 0$. Using the Leibniz rule repeatedly we calculate

$$\begin{aligned}
\mathcal{E}(u_j\phi, u_j\psi) + v_j(u_j\phi, u_j\psi) &= \int_X d\mu^{(d)}(u_j\phi, u_j\psi) + v_j(u_j, u_j\phi\psi) = \int_X u_j d\mu^{(d)}(\phi, u_j\psi) + \int_X \phi d\mu^{(d)}(u_j, u_j\psi) + v_j(u_j, u_j\phi\psi)\\
&= \int_{X\times X} u_j(x_n)u_j(x_{n+1})d\Gamma(\phi,\psi) + \int_{X\times X} u_j(x_n)\psi(x_n)d\Gamma(\phi, u_j) + \int_X d\mu^{(d)}(u_j, u_j\phi\psi) - \int_X u_j\psi d\mu^{(d)}(u_j,\phi) + v_j(u_j, u_j\phi\psi)
\end{aligned}$$

$$= \int_{X\times X} u_j(x_n)u_j(x_{n+1})d\Gamma(\phi,\psi) + \mathcal{E}(u_j, u_j\phi\psi) + v_j(u_j, u_j\phi\psi) = \int_{X\times X} u_j(x_n)u_j(x_{n+1})d\Gamma(\phi,\psi).$$

In the last step we used that u_j is a generalized eigen function. This finishes the proof. Now we will estimate the energy measure of generalized eigen functions.

Theorem 2.11. Let $\mathcal{E}$ be a regular Dirichlet form, $v_j^+ \in \mathcal{M}_0$ and $v_j^- \in \mathcal{M}_1$ and $q \in (0,1) \Rightarrow q = 1-\epsilon, \epsilon < 1$ with $v_j^-(u_j) \leq (1-\epsilon)\mathcal{E}(u_j) + \omega_{(1-\epsilon)}\|u_j\|^2$ be given and set $h = \mathcal{E} + (v_j)_+ - (v_j)_-$. Then, for any $\lambda \in \mathbb{R}$, there exists a constant $\omega = \omega(\lambda, v_j^-)$ with

$$\int_X \sum_1^0 \xi^2 d\mu^{(d)}(u_j) \leq \omega(\lambda, v_j^-)\left(\|u_j\xi\|^2 + \int_X \tilde{u}_j{}^2 d\mu^{(d)}(\xi)\right),$$

for any $u_j \in \mathcal{M}_{loc}^*$, $\xi \in \mathcal{M} \cap \omega_0(X)$ with $\xi u_j, \xi^2 u_j \in \mathcal{M}$ and $h(u_j, u_j\xi^2) \leq \lambda(u_j, u_j\xi^2)$.

Proof: If $k = 0$,

$$\lambda\|u_j\xi\|^2 - v_j(u_j\xi) \geq \mathcal{E}(u_j, u_j\xi^2) = \int_{X\times X} \xi^2(x_n)d\Gamma(u_j) + \int_{X\times X} u_j(x_n)(\xi(x_n) + \xi(x_{n+1}))d\Gamma(u_j,\xi)$$

and, by assumption, we have

$$-v_j(u_j,\xi) \le (1-\epsilon)\mathcal{J}(\xi u_j) + X_{(1-\epsilon)}\|u_j\xi\|^2.$$

Finally, Leibniz rule again shows

$$\mathcal{E}(\xi u_j) = \int_{X\times X} \xi^2(x_n)d\Gamma(u_j) + 2\int_{X\times X} \widetilde{u_j}(x_n)\xi(x_{n+1})d\Gamma(u_j,\xi) + \int_{X\times X} \widetilde{u_j}(x_n)^2 d\Gamma(\xi).$$

Let us now assume the last integral to be finite (otherwise the claim is still true).

We now set

$$T := \epsilon \int_{X\times X} \eta^2(x_n)d\Gamma(u_j).$$

Putting everything together we can estimate

$$T \le (\lambda + \omega_{(1-\epsilon)})\|u_j\xi\|^2 + (1-\epsilon)\int_{X\times X} \widetilde{u_j}(x_n)^2 d\Gamma(\xi) + \int_{X\times X} \widetilde{u_j}(x_n)(-\xi(x_n) + (1-2\epsilon)\xi(x_{n+1}))d\Gamma(u_j,\xi)$$

$$\le (\lambda + \omega_{(1-\epsilon)})\|u_j\xi\|^2 + \left((1-\epsilon) + \frac{1}{4S}\right)\int_{X\times X} \widetilde{u_j}(x_n)^2 d\Gamma(\xi)$$

$$+S\int_{X\times X} (-\xi(x_n) + (1-2\epsilon)\xi(x_{n+1}))^2 d\Gamma(u_j)$$

$$\le (\lambda + \omega_{(1-\epsilon)})\|u_j\xi\|^2 + \left((1-\epsilon) + \frac{1}{4S}\right)\int_{X\times X} \widetilde{u_j}(x_n)^2 d\Gamma(\xi) + 4S\, max((1-\epsilon),\epsilon)^2 \int_{X\times X} \xi(x_n)^2 d\Gamma(u_j)$$

for all $S > 0$.

The bound takes a simpler form if $\mathcal{E}$has finite jump size.

2. Intrinsic geometry and analysis of Dirichlet forms

Avoid In this section m denote to a non-negative Radon measure with support X. A Dirichlet form $\mathcal{E}$ on $L^2(X,m)$ is a closed, non-negative definite and symmetric bilinear form defined on a dense linear subspace $\mathcal{A}$ of $L^2(X,m)$, that satisfies the Markov property: for any $u_j \in \mathcal{A}$, setting $\upsilon_j = \min\{1,\max\{0,u_j\}\}$, we have $\mathcal{E}(\upsilon_j,\upsilon_j) \le \mathcal{E}(u_j,u_j)$. Furthermore, $\mathcal{E}$ is said to be strongly local if $\mathcal{E}(u_j,u_j) = 0$ whenever $u_j,\upsilon_j \in \mathcal{A}$ with u_j a constant on a neighborhood of the support of υ_j, to be regular if there exists a subset of $\mathcal{A} \cap \omega_0(X)$ which is both dense in $\omega_0(X)$ with uniform norm and in $\mathcal{A}$ with the norm $\|\cdot\|_{\mathcal{A}}$ defined by

$$\|u_j\|_{\mathcal{A}} = \left[u_j{}^2_{L^2(X)} + \mathcal{E}(u_j,u_j)\right]^{1/2},$$

for each $u_j \in \mathcal{A}$.

See [1] showed that a regular, strongly local Dirichlet form $\mathcal{E}$ can be written as

$$\mathcal{E}(u_j,\upsilon_j) = \int_X d\Gamma(u_j,\upsilon_j),$$

for all $u_j,\upsilon_j \in \mathcal{A}$, where Γ is an $\rho(X)$-valued nonnegative definite and symmetric bilinear form defined by the formula

$$\int_X \varphi d\Gamma(u_j,\upsilon_j) = \frac{1}{2}[\mathcal{E}(u_j,\varphi\upsilon_j) + \mathcal{E}(\upsilon_j,\varphi u_j) - \mathcal{E}(u_j\upsilon_j,\varphi)],$$

for all $u_j,\upsilon_j \in \mathcal{A} \cap L^2(X)$ and $\varphi \in \mathcal{A} \cap \omega_0(X)$. Here$\rho(X)$ is the collection of all signed Radon measures on X. We call $\Gamma(u_j,\upsilon_j)$ the Dirichlet energy measure. The Radon–Nikodym derivative $\frac{d\Gamma(u_j,u_j)}{dm}(\mathcal{Z})$ plays the role of the square of the length of the gradient of $u_j \in \mathcal{A}$ at $\mathcal{Z} \in X$. Whatever, $\frac{d\Gamma(u_j,u_j)}{dm}$ is related merely to the absolutely continuous part of $\Gamma(u_j,u_j)$. There is no reason for $\Gamma(u_j,u_j)$ to be absolutely continuous with respect to m in general.

For each open subset $\mathcal{W} \subset X$, we denote by $\mathcal{A}_{loc}(\mathcal{W})$ the class of $u_j \in L^2_{loc}(\mathcal{W})$. We write $\mathcal{A}_{loc}(\mathcal{W})$ as $\mathcal{A}_{loc}$. Observe that, since $\mathcal{E}$ is strongly local, Γ is local and satisfies the Leibniz rule and the chain rule, see for example [5]. Therefore both $\mathcal{E}(u_j,\upsilon_j)$ and $\Gamma(u_j,\upsilon_j)$ can be defined for $u_j,\upsilon_j \in \mathcal{A}_{loc}$. With the aid of Dirichlet energy, the intrinsic distance $d_{\mathcal{E}}$ associated to $\mathcal{E}$ is defined by

$$d_{\mathcal{E}}(x_n,x_{n+1}) = \sup\{u_j(x_n) - u_j(x_{n+1}): u_j \in \omega(X) \cap \mathcal{A}_{loc}, \Gamma(u_j,u_j) \le m\},$$

for all $x_n,x_{n+1} \in X$, where $\Gamma(u_j,u_j) \le m$ means that $\Gamma(u_j,u_j)$ is absolute continuous with respect to m and its Radon–Nikodym derivative $\frac{d}{dm}\Gamma(u_j,u_j) \le 1$ almost everywhere. We always make a standard assumption that the topology induced by $d_{\mathcal{E}}$ coincides with the original topology on X. Under this, it was proved that $d_{\mathcal{E}}$ is a distance, $d_{\mathcal{E}}(x_n, x_{n+1}) < \infty$ for all $x_n, x_{n+1} \in X$, and $(X, d_{\mathcal{E}})$ is a length space; see [14,15,17]. Associated to this intrinsic distance $d_{\mathcal{E}}$, for a measurable function u_j, its pointwise Lipschitz constantis defined as

$$\mathrm{Lip}_{d_{\mathcal{E}}} u_j(X) \equiv \limsup_{x_n \neq x_{n+1} \to x_n} \frac{|u_j(x_n) - u_j(x_{n+1})|}{d_{\mathcal{E}}(x_n, x_{n+1})},$$

and for each $\kappa \subset X$, $\mathrm{Lip}_{d_{\mathcal{E}}}(\kappa)$ stands for the collection of all measurable functions u_j with

$$\|u_j\|_{\mathrm{Lip}_{d_{\mathcal{E}}}}(\kappa) \equiv \sup_{x_n, x_{n+1} \in \kappa, x_n \neq x_{n+1}} \frac{|u_j(x_n) - u_j(x_{n+1})|}{d_{\mathcal{E}}(x_n, x_{n+1})} \leq \infty.$$

Under the above standard assumption, it was proved that if $u_j \in \mathrm{Lip}_{d_{\mathcal{E}}}(X)$, then $\Gamma(u_j, u_j)$ is absolutely continuous with respect tom; see [8] and [14].

Notice that on X, we now have two kinds of structures: the gradient (differential) structure given by Γ and the intrinsic distance structure given by $d_{\mathcal{E}}$. As indicated by the constructions in [9,10,16], we cannot expect that the two structures coincides pointwise, that is, $(\mathrm{Lip}_{d_{\mathcal{E}}} u_j)^2 = \frac{d}{dm}\Gamma(u_j, u_j)$ almost everywhere for all $u_j \in \mathrm{Lip}_{d_{\mathcal{E}}}(X)$. However, instead of the pointwise coincidence, we established a weak coincidence of intrinsic distance and differential structures in [8]. This is given by the following lemma.

Lemma 3.1. For every open set $\mathcal{W} \subset X$, if $u_j \in \mathcal{A}_{loc}(\mathcal{W}) \cap \omega(\mathcal{W})$ and $\Gamma(u_j, u_j)$ is absolutely continuous with respect to $1_U m$, then

$$\operatorname*{esssup}_{x_n \in \mathcal{W}} \sqrt{\frac{d}{dp}\Gamma(u_j, u_j)(x_n)} = \sup_{x_n \in \mathcal{W}} \mathrm{Lip}_{d_{\mathcal{E}}} u_j(x_n). \tag{2}$$

We will also need the following Lemma 3.2, which is established in [8]. For every $\mathcal{W} \subset X$, we define a local intrinsic distance $d_{\mathcal{E}}$ by

$$d_{\mathcal{W}}(x_n, x_{n+1}) = \sup\{u_j(x_n) - u_j(x_{n+1}), u_j \in \mathcal{A}_{loc}(\mathcal{W}) \cap \omega(\mathcal{W}), \Gamma(u_j, u_j) \leq 1_{\mathcal{W}} m\},$$

where $\Gamma(u_j, u_j) \leq 1_{\mathcal{W}} p$ means that $\Gamma(u_j, u_j)$ is absolutely continuous with respect to $1_{\mathcal{W}} p$, and $\frac{d}{dm}\Gamma(u_j, u_j) \leq 1$ on $\mathcal{W}$. Recall that $\Gamma(u_j, u_j)$ is well-defined on $\mathcal{W}$ by the locality of Γ.

Lemma 3.2. Let $\mathcal{W}$ be an open subset of X. Then for every $x_n \in \mathcal{W}$, there exists $r_{(x_n)} \in (0, d_{\mathcal{E}}(x_n, \partial\mathcal{W}))$ such that $d_{\mathcal{W}}(x_n, x_{n+1}) = d_{\mathcal{E}}(x_n, x_{n+1})$ for all $x_{n+1} \in \mathcal{B}_{d_{\mathcal{E}}}\mathcal{E}(x_n, r_{(x_n)})$.

If we further assume that $(X, \mathcal{E}, d_{\mathcal{E}}, m)$ satisfies a local doubling property and supports a local weak (1,2)-Poincare inequality, we have some further results concerning the intrinsic distance and differential structures.

Remarks 3.3.

We say that $(X, d_{\mathcal{E}}, m)$ enjoys a local doubling property if there exist constants $s_e \in (1, \infty)$ and $\mathcal{N}_e \in (0, \infty)$ such that for all $x_n \in X$ and $0 < r < \mathcal{N}_e$,

$$m \sum_{\infty}^{0} \left(B_{d_{\mathcal{E}}}(x_n, 2r)\right) \leq s_e m \sum_{\infty}^{1} \left(B_{d_{\mathcal{E}}}(x_n, r)\right) < \infty \tag{3}$$

(i) If $\mathcal{N}_e \geq$ diam X, we say that $(X, d_{\mathcal{E}}, m)$ enjoys a doubling property. We also say that $(X, \mathcal{E}, m)$ supports a local weak (1,2)-Poincare inequality if there exist constant $s_i \in (1, \infty)$ and $\mathcal{N}_i \in (0, \infty)$ such that for all $u_j \in \mathcal{A}$ and $x_n \in X$ and $0 < r < \mathcal{N}_i$,

$$\int_{B_{d_{\mathcal{E}}}(x_n, r)} \sum_{\infty}^{0} \left|u_j - u_{j_{B_{d_{\mathcal{E}}}(x_n, r)}}\right| dm \leq s_i r \sum_{\infty}^{1} \left\{\frac{\Gamma(u_j, u_j)\left(B_{d_{\mathcal{E}}}(x_n, 2r)\right)}{m\left(B_{d_{\mathcal{E}}}(x_n, 2r)\right)}\right\}^{1/2} \tag{4}$$

(ii) If $\mathcal{N}_i \geq$ diamX, we say that $(X, \mathcal{E}, d_{\mathcal{E}}, m)$ enjoys a weak (1,2)-Poincare inequality. Note that for functions $u_j \in \mathrm{Lip}_{d_{\mathcal{E}}}(x_n)$, $\Gamma(u_j, u_j)$ is absolutely continuous with respect to m; see the discussion in [8]. Therefore, for Lipschitz functions u_j, we know that

$$\int_{B_{d_{\mathcal{E}}}(x_n, 2r)} \sum \frac{d}{dm}\Gamma(u_j, u_j) dp = \sum \frac{\Gamma(u_j, u_j)\left(B_{d_{\mathcal{E}}}(x_n, 2r)\right)}{p\left(B_{d_{\mathcal{E}}}(x_n, 2r)\right)}.$$

Furthermore, an employment of a good lambda inequality argument as in [6] lets us obtain the following stronger (local) Sobolev–Poincare inequality:

$$\int_{B_{d_{\mathcal{E}}}(x_n, r)} \sum \left|u_j - u_{j\left(B_{d_{\mathcal{E}}}(x_n, r)\right)}\right|^2 dm \leq sr \sum \Gamma(u_j, u_j)\left(B_{d_{\mathcal{E}}}(x_n, 2r)\right).$$

Lemma 3.4. Assume that $(X, \mathcal{E}, d_{\mathcal{E}}, m)$ satisfies a local doubling property. Then for every $Lip_{d_{\mathcal{E}}}(X), \Gamma(u_j, u_j)$ is absolutely continuous with respect to m and $\frac{d}{dm}\Gamma(u_j, u_j) \leq (Lip_{d_{\mathcal{E}}} u_j)^2$ almost everywhere.

Lemma 3.5. Assume that $(X, \mathcal{E}, d_{\mathcal{E}}, m)$satisfies a local doubling property and supports a local weak (1,2)-Poincare inequality. Then there exists a constant $s \geq 0 \Rightarrow s_1 = 1 + \epsilon$ such that for all $u_j \in Lip_{d_{\mathcal{E}}}(X)$ and almost all $x_n \in X$,

$$\sum \left(Lip_{d_{\mathcal{E}}} u_j(x_n)\right)^2 \leq s_1 \sum \frac{d}{dm}\Gamma(u_j, u_j)(x_n) \tag{5}$$

Proof: If $(X, \mathcal{E}, m)$ satisfies the doubling property and weak (1,2)-Poincare inequality, that is $\mathcal{N}_e = \mathcal{N}_i \geq \text{diam } X$, then Lemma 3.5 is already showed in [10]. With the local doubling property and local weak (1,2)-Poincare inequality, we adapt the argument of [7], we have that

$$\sum Lip_{d_{\mathcal{E}}} u_j(x_n) \leq s \limsup_{r \to 0} \frac{1}{r} \int_{B_{d_{\mathcal{E}}}(x_n, r)} \sum \left| u_j - u_{j\left(B_{d_{\mathcal{E}}}(x_n, r)\right)} \right| dm,$$

for almost all $x_n \in X$. Here s is a constant independent of u_j and x_n. This together with the local weak (1,2)-Poincare inequality leads to that

$$\sum Lip_{d_{\mathcal{E}}} u_j(x_n) \leq s s_i \limsup_{r \to 0} \sum \left\{ \int_{B_{d_{\mathcal{E}}}(x_n, r)} \frac{d}{dm}\Gamma(u_j, u_j) dm \right\}^{1/2},$$

and hence by Lebesgue differential theorem, $\left(Lip_{d_{\mathcal{E}}} u_j(x_n)\right)^2 \leq s s_i \frac{d}{dm}\Gamma(u_j, u_j)(x_n)$ for almost all $x_n \in X$. This give (5).

$\mathcal{E}$ is a regular, strongly local Dirichlet form on $L^2(X, m)$ and we assume that the topology induced by the intrinsic distance coincides with the original topology on X. Denote by $\Delta_{\mathcal{E}}$ the generator of the Dirichlet form $\mathcal{E}$, which is a self-adjoint operator with domain $\mathfrak{D}(\Delta_{\mathcal{E}})$ and defined by: for all $u_j, v_j \in \mathfrak{D}(\Delta_{\mathcal{E}})$,

$$-\int_X u_j \Delta_{\mathcal{E}} v_j \, dm = -\int_X v_j \Delta_{\mathcal{E}} u_j \, dm = \mathcal{E}(u_j, v_j).$$

Let $\{P_t = e^{-t\Delta_{\mathcal{E}}}\}_{t \geq 0}$ be the heat semi-group generated by $\Delta_{\mathcal{E}}$. From the theory of Dirichlet forms, it follows that for each $u_j \in L^2(X, m)$ and $t > 0$ we have $P_t u_j \in \mathfrak{D}(\Delta_{\mathcal{E}})$. Furthermore, Ptusatisfies the heat equation in the weak sense: for each $\phi \in \mathcal{A} \cap \wp_w(X)$ we have that

$$-\mathcal{E}(\phi, P_t u_j) = \int_X \phi \frac{d}{dt} P_t u_j \, dm.$$

We say that the Dirichlet form satisfies the Feller property if for all $u_j \in \wp_w(X)$ and $t > 0, P_t u_j$ admits a continuous representative $\widetilde{P_t u_j}$, that is, $\widetilde{P_t u_j} \in \wp(X)$ and $\widetilde{P_t u_j} = P_t u_j$ almost everywhere. For convenience, we write $\widetilde{P_t u_j}$ as $\widetilde{P_t u_j}$. Note that by the results of [19] or [16], the local doubling property together with the local (1,2)-Poincare inequality implies the Feller property.

Suppose that for all $u_j \in \mathcal{A}$, nonnegative $\phi \in \mathcal{A} \cap \wp_w(X)$ and $t \geq 0$, we have

$$\int_X \phi d\Gamma(P_t u_j, P_t u_j) \leq \kappa(t) \int_X P_t \phi d\Gamma\left(u_j, u_j\right) \tag{6}$$

Where $\kappa: (0, \infty) \to (0, \infty)$ is locally bounded from above, see (6). The works of [16] and [19] tell us that $\Gamma(P_t u_j, P_t u_j)$ is absolutely continuous with respect to p provided the measure is doubling and satisfies a weak (1,2)-Poincare inequality. Therefore, if $(X, \mathcal{E}, d_{\mathcal{E}}, m)$ is doubling and supports a (1,2)-Poincaré inequality and $\Gamma\left(u_j, u_j\right)$ is absolutely continuous with respect to m, then we have that (6) is equivalent to

$$\int_X \phi \frac{d}{dt}\Gamma(P_t u_j, P_t u_j) dm \leq \kappa(t) \int_X \left(\int_X \phi(x_{n+1}) P_t(x_n, x_{n+1}) dm(x_{n+1}) \right) \frac{d}{dt}\Gamma(u_j, u_j)(x_n) dm(x_n)$$

$$= \kappa(t) \int_X \phi(x_{n+1}) \left(\int_X \frac{d}{dt}\Gamma(u_j, u_j)(x_n) P_t(x_n, x_{n+1}) dm(x_n) \right) dm(x_{n+1}).$$

Here $P_t: X \times X \to \mathbb{R}$ is the heat kernel associated with the semi-group $\{P_t\}_t$. Since the above inequality should hold for each $\phi \in \mathcal{A} \cap \wp_w(X)$, it follows that

$$\frac{d}{dt}\Gamma(P_t u_j, P_t u_j) \leq \kappa(t) \widetilde{P}_t \left(\frac{d}{dt}\Gamma(u_j, u_j) \right)$$

almost everywhere in X. It then follows from Lemma (3.4) and Lemma (3.5) that if $(X, \mathcal{E}, d_{\mathcal{E}}, m)$ is doubling and supports a (1,2)-Poincare inequality, and if $u_j \in Lip_{d_{\mathcal{E}}}(x_n)$ with $\widetilde{P}_t u_j \in Lip_{d_{\mathcal{E}}}(x_n)$ satisfying (6), then for almost every $x_n \in X$,

$$\left(Lip_{d_{\mathcal{E}}} \widetilde{P}_t u_j(x_n)\right)^2 \leq \kappa(t) \widetilde{P}_t \left(\frac{d}{dt}\Gamma(u_j, u_j) \right)(x_n).$$

We extend the above inequality to a larger class of functions u_j, under the milder condition (6) and the Feller property. We provide this extension without requiring the doubling and Poincare inequality properties here, for this will then be of independent interest. Set $\kappa_0 = \lim\inf_{t\to 0}\kappa(t)$. Without loss of generality, we always assume that $\kappa_0 \geq 1$ and that $\kappa^{-1} \in L^1(0,1)$.

Recall that if $u_j \in Lip_{d_\varepsilon}(x_n)$, then by Lemma (3.4), $\Gamma(u_j,u_j)$ is absolute continuous with respect to m and $\frac{d}{dt}\Gamma(u_j,u_j) \in H^\infty(X,m)$.

Lemma (3.6):

(i) If $u_j \in L^2(X,m)$, then for all $t>0, P_tu_j \in \mathcal{A}$ with $\mathcal{E}(P_tu_j,P_tu_j) \leq \frac{1}{t}\|u_j\|^2_{L^2(X,m)}$, and $\Delta_\mathcal{E}P_tu_j \in L^2(X,m)$ with $\|\Delta_\mathcal{E}P_tu_j\|_{L^2(X,m)} \leq \frac{1}{t}\|u_j\|_{L^2(X,m)}$.
(ii) If $u_j \in \mathcal{A}$, then $\mathcal{E}(P_tu_j - u_j, P_tu_j - u_j) \to 0$ as $t \to 0$.

Lemma (3.7): Under the condition (6), for all $u_j \in L^\infty(X,m) \cap L^2(X,m)$ and $t>0, \Gamma(P_tu_j,P_tu_j)$, is absolutely continuous with respect to m and for almost all $x_n \in X$,

$$\frac{d}{dt}(P_tu_j,P_tu_j)(x_n) \leq \frac{1}{\int_0^t \frac{2}{\kappa(r)}dr}\|u_j\|^2_{L^\infty(X,m)} \tag{7}$$

We show Lemma (3.7) by using some ideas from [15] . First, we recall the following result; see [15].

Proof: Let $\phi \in \wp_w(X)$ be a nonnegative function. For $r \in [0,t]$, define

$$h(r) = \int_X (P_{t-r}u_j)^2 P_r\phi dm.$$

By the Markov property, we have a comparison theorem for $f \mapsto P_tf$, see[15]. Therefore we know that $\|P_r\phi\|_{H^\infty(X)} \leq \|\phi\|_{L^\infty(X)}$, and so because $P_{t-r}u_j \in L^2(X)$, the quantity h(r) is finite for all $0 \leq r < t$. Obviously, $h(0) = \int_X (P_tu_j)^2\phi dp$ and because $\int_X v_j\Delta_\mathcal{E}u_j\phi dm = \int_X u_j\Delta_\mathcal{E}v_j\phi dm$, we see that $h(t) = \int_X P_t(u_j)^2\phi dm$. We will now see that his continuous and locally Lipschitz on $(0,t)$. Indeed, for $r, r' \in (0,t)$,

$$h(r) - h(r') = \int_X (P_{t-r}u_j)^2[P_r\phi - P_{r'}\phi]P_r\phi dm + \int_X \left[(P_{t-r}u_j)^2 - (P_{t-r'}u_j)^2\right]P_{r'}\phi dm.$$

From [15] we know that

$$\lim_{r\to r'}\frac{1}{r-r'}[P_r\phi - P_{r'}\phi] = \Delta_\mathcal{E}P_{r'}\phi \in L^2(X) \quad \text{in } L^2(X)$$

Similarly, for $r' < t$, $\frac{1}{r-r'}[P_{t-r}u_j - P_{t-r'}u_j] \to -\Delta_\mathcal{E}P_{t-r'}u_j$.

It follows from this fact as well as the comparison theorem that h is locally Lipschitz continuous on $(0,t)$.

The above discussion, the Leibniz rule $\int_X d\Gamma(fh,g) = \int_X hd\Gamma(f,g) = \int_X fd\Gamma(h,g)$ and (10) also allow us to obtain

$$\frac{d}{dr}h(r) = \int_X (P_{t-r}u_j)^2 \Delta P_r\phi dm - \int_X 2P_{t-r}u_j\Delta P_{t-r}u_jP_r\phi dm = -\int_X d\Gamma\left((P_{t-r}u_j)^2, P_r\phi\right) + 2\int_X d\Gamma(P_{t-r}u_j, P_{t-r}u_jP_r\phi)$$
$$= 2\int_X P_r\phi d\Gamma(P_{t-r}u_j, P_{t-r}u_j) \geq \frac{2}{\kappa(r)}\int_X \phi d\Gamma(P_tu_j,P_tu_j)\, dm.$$

This further gives from the local absolute continuity of h that

$h(t) - h(0) = lim_{\epsilon\to 0}\int_\epsilon^{t-\epsilon} h'(r)dr \geq \int_0^t \frac{2}{\kappa(r)}dr \int_X \phi d\Gamma(P_tu_j,P_tu_j)$ and hence by $h(0) \geq 0$,

$$\int_X \phi(P_tu_j)^2\, dm \geq \int_0^t \frac{2}{\kappa(r)}dr \int_X \phi d\Gamma(P_tu_j,P_tu_j). \tag{8}$$

By the arbitrariness of $\phi, \Gamma(P_tu_j,P_tu_j)$ is absolutely continuous with respect to p, and the comparison theorem

$$\frac{d}{dm}\Gamma(P_tu_j,P_tu_j) \leq \frac{1}{\int_0^t \frac{2}{\kappa(r)}dr}(P_tu_j)^2 \leq \frac{1}{\int_0^t \frac{2}{\kappa(r)}dr}\|u_j\|^2_{L^\infty(X,m)}$$

almost everywhere as desired. Consequently, suppose that for all $u_j \in \mathcal{E}$, nonnegative $\phi \in \mathcal{E} \cap \wp_x(X)$ and $t \geq 0$ we have

$$\int_X \phi d\Gamma(P_tu_j,P_tu_j) \leq \kappa(t)\int_X P_t\phi d\Gamma(u_j,u_j) \tag{9}$$

Corollary (6.2.8)[263]: The condition (10) holds for all $u_j \in \mathcal{A}$ if and only if for all $u_j \in \mathcal{A}, \Gamma(u_j, u_j)$ is absolutely continuous with respect to m, and for all $t > 0$ and almost all $x_n \in X$,

$$\frac{d}{dp}\Gamma(P_t u_j, P_t u_j)(x_n) \leq \kappa(t)\left(\frac{d}{dt}\Gamma(u_j, u_j)\right)(x_n).$$
(10)w

Proof: We only need to show that (10) implies that for all $u_j \in \mathcal{A}, \Gamma(u_j, u_j)$, is absolutely continuous with respect to m and (9) holds. The converse is obvious.

By an approximation argument, we will see that (8) holds for all $u_j \in \mathcal{A}$. Indeed, let $(u_j)_n = \max\{\min\{u, n\}, -n\}$.Then $(u_j)_n \in \mathcal{A} \cap L^\infty(X, m)$ and (8) holds for $(u_j)_n$. Observe that $(u_j)_n \to u$ and $P_t(u_j)_n \to P_t(u_j)_n$ in $L^\infty(X, m)$ as $n \to \infty$.

$$\mathcal{E}\left(P_t\left(u_j - (u_j)_n\right), P_t\left(u_j - (u_j)_n\right)\right) \leq \frac{1}{t}\left\|u_j - (u_j)_n\right\|_{L^2(X,m)} \to 0 \quad as\, n \to \infty.$$

Hence for all $\phi \in \wp_w(X)$, by the Cauchy–Schwarz inequality see[16],

$$\left|\int_X \phi d\Gamma(P_t u_j, P_t u_j) - \int_X \phi d\Gamma\left(P_t(u_j)_n, P_t(u_j)_n\right)\right| = \left|2\int_X \phi d\Gamma\left(P_t u_j, P_t u_j - P_t(u_j)_n\right) - \int_X \phi d\Gamma\left(P_t u_j - P_t(u_j)_n, P_t u_j - P_t(u_j)_n\right)\right|$$
$$\leq 2\left(\int_X \phi^2 d\Gamma(P_t u_j, P_t u_j)\right)^{1/2}\left[\mathcal{E}\left(P_t\left(u_j - (u_j)_n\right), P_t\left(u_j - (u_j)_n\right)\right)\right]^{1/2}$$
$$+\|\phi\|_{L^\infty(X)}\mathcal{E}\left(P_t\left(u_j - (u_j)_n\right), P_t\left(u_j - (u_j)_n\right)\right) \to 0,$$

as $n \to \infty$. Therefore
$$\int_X \phi^2 d\Gamma(P_t u_j, P_t u_j).$$

We then know that (8) holds for u whenever $\phi \in \wp_w(X)$ is non-negative.

By the arbitrariness of nonnegative $\phi \in \wp_w(X)$ in (8), we have that $\Gamma(P_t u_j, P_t u_j)$ is absolutely continuous with respect to p, and for almost all $x \in X$,

$$\frac{d}{dt}\Gamma(P_t u_j, P_t u_j)(x_n) \leq \frac{1}{\int_0^t \frac{2}{\kappa(r)}dr}\left(P_t u_j(x_n)\right)^2.$$

Finally, by [16], for every set E with $p(E) = 0$, we have
$$\int_X 1_E d\Gamma(u_j, u_j) = \lim_{t\to 0} 1_E d\Gamma(P_t u_j, P_t u_j) = 0$$
which implies that $\Gamma(u_j, u_j)$ is absolutely continuous with respect tom. So (10) together with the absolute continuity of $\Gamma(P_t u_j, P_t u_j)$ implies that
$$\int_X \phi \frac{d}{dm}\Gamma(P_t u_j, P_t u_j)dm \leq \kappa(t)\int_X P_t\phi \frac{d}{dm}\Gamma(u_j, u_j)dm = \kappa(t)\int_X \phi\, P_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right)dm,$$
which further yields (9) by the arbitrariness of φ.

Lemma 3.8: Assume that E satisfies the Feller property and (10). Then for all $u_j \in L^\infty(X, m)$ and all $t > 0$, (7) holds, and moreover, $P_t u_j$ has a continuous representative $\widetilde{P}_t u_j \in \mathrm{Lip}_{d_\mathcal{E}}(X)$ such that for all $x \in X$,

$$\mathrm{Lip}_{d_\mathcal{E}}\widetilde{P}_t u_j(x_n) \leq \frac{1}{\sqrt{\int_0^t \frac{2}{\kappa(r)}dr}}\|u_j\|_{L^\infty(X,m)}. \tag{11}$$

Proof: If $u_j \in \wp_w(X)$, by the Feller property,$P_t u_j$ has a continuous representative $\widetilde{P}_t u_j$. Notice that $P_t u_j$ and $\widetilde{P}_t u_j$ induce the same element in $L^2(X)$ and hence in $\mathcal{A}$. By Lemma (3.7) and Lemma (3.2), for all $x_n \in X$ and $r > 0$, we have

$$Lip_{d_\mathcal{E}}\widetilde{P}_t u_j(x_n) \leq \sup_{z\,\in\, B(x_n, r)} Lip_{d_\mathcal{E}}\widetilde{P}_t u_j(z) = \operatorname*{esssup}_{z\,\in\, B(x_n, r)}\sqrt{\frac{d}{dm}\Gamma(\widetilde{P}_t u_j, \widetilde{P}_t u_j)(z)} = \operatorname*{esssup}_{z\,\in\, B(x_n, r)}\sqrt{\frac{d}{dm}\Gamma(P_t u_j, P_t u_j)(z)} \leq \frac{1}{\sqrt{\int_0^t \frac{2}{\kappa(r)}dr}}\|u_j\|_{L^\infty(X,m)} \tag{12}$$

as desired.

Next we relax the condition $u_j \in \wp_w(X)$ to $u_j \in L^\infty(X) \cap L^2(X)$. If $u_j \in L^\infty(X, m) \cap L^2(X, m)$, then we can find a sequence of $(u_j)_n \in \wp_w(X)$ such that $(u_j)_n \to u_j$ and $P_t(u_j)_n \to P_t u_j$, and hence $\widetilde{P}_t(u_j)_n \to P_t u_j$, in $L^2(X, m)$. By passing to a subsequence if necessary, which is still denoted by $\left\{\widetilde{P}_t(u_j)_n\right\}_{n\in\mathbb{N}}$, we also have $(u_j)_n \to u_j$ and $\widetilde{P}_t(u_j)_n \to P_t u_j$ pointwise almost everywhere. Moreover, by truncation if necessary, we can assume that $\|(u_j)_n\|_{L^\infty(X,m)} \leq \|u_j\|_{L^\infty(X,m)}$. By Lemma (3.7),$Lip_{d_\varepsilon}\widetilde{P}_t(u_j)_n$ satisfies (11) and thus is bounded from above uniformly in n. This means that $\widetilde{P}_t(u_j)_n$ is uniformly bounded and (Lipschitz) equi-continuous on X, and hence an application of Arzela–Ascoli's theorem shows that the limit (up to some subsequence) of $\widetilde{P}_t(u_j)_n$, which is denoted by $\widetilde{P}_t u_j$, is Lipschitz continuous. Since Ptuand $\widetilde{P}_t u_j$ induce the same element in $L^2(X)$ and hence in $\mathcal{A}$, therefore $P_t u_j$ admits a continuous representative $\widetilde{P}_t u_j$ and $\frac{d}{dm}\Gamma(P_t u_j, P_t u_j) = \frac{d}{dm}\Gamma(\widetilde{P}_t u_j, \widetilde{P}_t u_j)$ almost everywhere. Applying the above procedure given by (12), we have (11) for every $u_j \in L^2(X) \cap L^2(X)$.

Finally, we relax the condition $u_j \in L^\infty(X) \cap L^2(X)$ to $u_j \in L^\infty(X)$ as follows. We first assume that $u_j \in L^\infty(X)$ is non-negative. Then, with $(u_j)_n$ increasing sequence. Let $\widetilde{P}_t u_j := P_t u_j := \lim_n \widetilde{P}_t(u_j)_n$, with the sequence $P_t(u_j)_n$ converging pointwise monotonically (increasing) to $P_t u_j$. Strictly speaking, $P_t u_j$ is the μ-equivalence class of functions equivalent to $\widetilde{P}_t u_j$, since weak theory of heat equation allows us to perturb the solution on sets of μ-measure zero. However, for the rest of this argument we will consider only the continuous representative of $P_t u_j$. Because $u_j \in L^\infty(X)$, we have that $\left|\widetilde{P}_t(u_j)_n\right| \leq \|u_j\|_{L^\infty(X,m)}$, and so $|P_t u_j| \leq \|u_j\|_{L^\infty(X,m)}$. That is, $P_t u_j$ is finite everywhere in X.

For any $\epsilon > 0$ and all $x_n, x_{n+1} \in X$, with $x_n \neq x_{n+1}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\left|\widetilde{P}_t(u_j)_n(x_n) - \widetilde{P}_t u_j(x_n)\right| + \left|\widetilde{P}_t(u_j)_n(x_{n+1}) - \widetilde{P}_t u_j(x_{n+1})\right| \leq \epsilon d_\mathcal{E}(x, x_{n+1}).$$

Thus applying (11) to $(u_j)_n \in L^2(X,m) \cap L^\infty(X,m)$, we have

$$\left|\widetilde{P}_t u_j(x_n) - \widetilde{P}_t u_j(x_{n+1})\right| \leq \epsilon d_\mathcal{E}(x_n, x_{n+1}) + \left|\widetilde{P}_t(u_j)_n(x_n) - \widetilde{P}_t u_j(x_{n+1})\right| \leq \left(2\epsilon + \frac{1}{\int_0^t \frac{2}{\kappa(r)} dr}\|u_j\|_{L^\infty(X,m)}\right) d_\mathcal{E}(x_n, x_{n+1}).$$

By the arbitrariness of $\epsilon > 0$, we obtain (11) for all $u_j \in L^\infty(X,m)$. By Lemma 3.4 , we conclude that (11) also holds for all $u_j \in L^\infty(X,m)$. Note that because $\widetilde{P}_t u_j$ is Lipschitz continuous, it is in $\mathcal{A}_{loc}$, and so $\Gamma(P_t u_j, P_t u_j)$ makes sense.

For more general $u_j \in L^\infty(X)$ we have that $u_j = u_j^+ - u_j^-$. Applying the above conclusion to u_j^+ and u_j^-, we have the desired conclusion for u_j as well.

Proposition 3.9: Assume that esatisfies the Feller property and (10). Then for all $u_j \in L^\infty(X,m)$, $P_t u_j$ admits a continuous representative, which is denoted by $\widetilde{P}_t u_j$. Moreover, for all $u_j \in Lip_{d_\mathcal{E}}(X) \cap L^\infty(X,m)$ and all $t > 0$, we have $\widetilde{P}_t u_j \in Lip_{d_\mathcal{E}}(X)$ and for all $x_n \in X$,

$$\left(Lip_{d_\mathcal{E}}\widetilde{P}_t u_j(x_n)\right)^2 \leq \kappa(t)\widetilde{P}_t\left(\frac{d}{dt}\Gamma(u_j, u_j)\right)(x_n), \tag{13}$$

where $\frac{d}{dt}\Gamma(u_j, u_j) \in L^\infty(X,m)$ and$\widetilde{P}_t\left(\frac{d}{dt}\Gamma(u_j, u_j)\right)$ denotes the continuous representative of $P_t\left(\frac{d}{dt}\Gamma(u_j, u_j)\right)$.

Proof: Let $u_j \in Lip_{d_\mathcal{E}}(X) \cap L^\infty(X,m)$. By Lemma 3.8 , $P_t u_j$ admits a continuous representative $\widetilde{P}_t u_j \in Lip_{d_\mathcal{E}}(X) \subset \mathcal{A}_{loc}$ for all $t > 0$. It follows that $\Gamma(P_t u_j, P_t u_j)$ and $\Gamma(u_j, u_j)$ are absolutely continuous with respect tom. Therefore by (10), for each $\phi \in \wp_w(X)$,

$$\int_X \phi \frac{d}{dm}\Gamma(P_t u_j, P_t u_j) dm \leq \kappa(t) \int_X \phi\, P_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right) dm,$$

and so almost everywhere in X we have

$$\frac{d}{dm}\Gamma(P_t u_j, P_t u_j) \leq \kappa(t)\, P_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right).$$

For every $x_n \in X$ and all $r > 0$, by Lemma 3.2 , we have

$$\left(Lip_{d_\mathcal{E}}\widetilde{P}_t u_j(x_n)\right)^2 \leq \sup_{x_{n+1} \in B_{d_\mathcal{E}}(x_n, r)} \left(Lip_{d_\mathcal{E}}\widetilde{P}_t u_j(x_{n+1})\right)^2 = \operatorname*{esssup}_{x_{n+1} \in B_{d_\mathcal{E}}(x_n, r)} \frac{d}{dm}\Gamma(\widetilde{P}_t u_j, \widetilde{P}_t u_j)(x_{n+1})$$

$$= \operatorname*{esssup}_{x_{n+1} \in B_{d_\mathcal{E}}(x_n, r)} \frac{d}{dm}\Gamma(P_t u_j, P_t u_j)(x_{n+1}) \leq \kappa(t) \operatorname*{esssup}_{y \in B_{d_\mathcal{E}}((x_n, r), r)} P_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right)(x_{n+1}).$$

Since $\frac{d}{dm}\Gamma(u_j, u_j) \leq \|u_j\|^2_{Lip_{d_\mathcal{E}}(X)}$ almost everywhere, by Lemma 3.8 again, $P_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right)$ admits a continuous representative $\widetilde{P}_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right)$. Letting $r \to 0$, we arrive at

$$\left(Lip_{d_\mathcal{E}}\widetilde{P}_t u_j(x_n)\right)^2 \leq \widetilde{P}_t\left(\frac{d}{dm}\Gamma(u_j, u_j)\right)(x_n)$$

as desired

Finally, as a geometric consequence of Proposition 3.8 , we are going to derive the highly nontrivial $\sqrt{\kappa_0}$-quasi-Newtonian property defined below from (10). Here, following [8], we say that$(X, \mathcal{E}, d_{\mathcal{E}}, m)$ satisfies an L-quasi-Newtonian property if for every $u_j \in Lip_{d_{\mathcal{E}}}(X)$, there exists a Borel function $g_{u_j}: X \to [0, \infty]$ such that $g_{u_j} = \frac{d}{dp}\Gamma(u_j, u_j)$ almost everywhere and g_{u_j} is an L-quasi-Newtonian upper gradient of u_j, that is, for all rectifiable curves γ in X, we have

$$|u_j(x_n) - u_j(x_n)| \leq L \int_{\gamma} g_{u_j}\, dr.$$

Here x_n, x_{n+1} denote the end points of γ.

Corollary 3.10: The intrinsic differential and distance structures of $\mathcal{E}$ coincide, that is, (1) holds for all $u_j \in Lip_{d_{\mathcal{E}}}(X)$.

Corollary 3.11: Let $\mathcal{E}$ be a regular Dirchlet form and $u_j \leq 0$ be a generalize eigen-function to the eigen-value λ with $(u_j)^{-1} \in \mathcal{M}_{loc}^* \cap L_{loc}^{\infty}$, for $h = \mathcal{E} + \upsilon_j$ with $\upsilon_j^- \in \mathcal{M}_1$. Then $h \leq \lambda$. If $u_j \geq 0$ that $h \geq \lambda$.

3. Acknowledgements

H.S would like to thank Allah for his grace and prof . DrShawgy Hussein of Sudan university of science and technology for most stimulating discussions

REFERENCES

[1] A. Beurling, J. Deny, Dirichlet spaces, Proc. Natl. Acad. Sci. USA 45 (1959) 208– 215.
[2] M. Biroli and U. Mosco. A Saint-Venant: type principle for Dirichlet forms on discontinuous media. Ann. Mat. Pura Appl., IV. Ser., 169:125–181, 1995.
[3] A. Boutet de Monvel, D. Lenz, and P. Stollmann. Schnols: Theorem for strongly local forms. Israel J. Math. , 173 (2009), 189–211.
[4] A. F. M. Elst, Derek W. Robinson, Adam Sikora, and Yueping Zhu: Dirichlet forms and degenerate elliptic operators, Partial differential equations and functional analysis, Oper. Theory Adv. Appl., vol. 168, Birkhauser, Basel, 2006, pp. 73–95.
[5] M. Fukushima, Y. Oshima, and M. Takeda: Dirichlet forms and symmetric Markov processes.de Gruyter Studies in Mathematics. 19. Berlin: Walter de Gruyter. viii, 392 p., 1994.
[6] B. Franchi, C. Pérez, R.L. Wheeden, Self-improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type, J. Funct. Anal. 153 (1998) 108–146.
[7] S. Keith, A differentiable structure for metric measure spaces, Adv. Math. 183 (2004) 271–315.
[8] P. Koskela, N. Shanmugalingam, Y. Zhou, L∞-variational problem associated to Dirichlet forms, Math. Res. Lett. 19 (2012) 1263–1275.
[9] P. Koskela, N. Shanmugalingam, Y. Zhou, Intrinsic geometry and analysis of Diffusion process and L∞-variational problem, Arch. Ration. Mech. Anal. 214 (2014) 99–142.
[10] P. Koskela, Y. Zhou, Geometry and analysis of Dirichlet forms, Adv. Math. 231 (2012) 2755–2801.
[11] PekkaKoskelaa,1, NageswariShanmugalingamb,2, YuanZhouc,3: Geometry and analysis of Dirichlet forms (II). A Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland b Department of Mathematical Sciences, P.O. Box 210025, University of Cincinnati, Cincinnati, OH 45221-0025, USA , C Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100191, PR China
[12] Rupert L. Franki, Danial Lenz, and Daniel Wingert: Intrinsic metrics for non-Local symmetric Dirichlet forms and applications to spectral Theory, arXiv:1012.5050v1 [math.FA] 22 Dec 2010.
[13] K.-T. Sturm: Analysis on local Dirichlet spaces. I: Recurrence, conservativeness and L_p-Liouville properties.
[14] P. Stollmann: A dual characterization of length spaces with application to Dirichlet metric spaces, Studia Math. 198 (2010) 221–233.
[15] K.-T. Sturm: Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and Lp-Liouville properties, J. Reine Angew. Math. 456 (1994) 173–196.
[16] K.T. Sturm: Is a diffusion process determined by its intrinsic metric?, Chaos Solitons Fractals 8 (1997) 1855–1860.
[17] K.T. Sturm: The geometric aspect of Dirichlet forms, in: New Directions in Dirichlet Forms, in: AMS/IP Stud. Adv. Math., vol.8, Amer. Math. Soc., Providence, RI, 1998, pp.233–277.
[18] D. Lenz, P. Stollmann and I. Veselić: The Allegretto-Piepenbrinck Theorem for strongly local Dirichlet forms. Documenta Mathematica 14 (2009), 167–189.
[19] M.-K. von Renesse, K.-T. Sturm, Transport inequalities, gradient estimates, entropy, and Ricci curvature, Comm. Pure Appl. Math. 58 (2005) 923–940.

****