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RESEARCH ARTICLE

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USING MOVING GRIDS METHOD FOR WATER DEPTH EVOLUTION STUDY IN THE COUPLED SYSTEM SAINT VENANT-EXNER EQUATION

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ABSTRACT

In this paper, the moving grids method based on the method of lines is used to simulate the Saint Venant-Exner coupled system where we focus on the water depth evolution. To illustrate the efficiency and accuracy property of the present method, we compare the computed solutions with a reference one obtained using a very fine mesh. The calculations give good agreement between reference solutions and numerical solutions.

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INTRODUCTION

Water needs require the recovery of surface water, which is why the construction of drainage channels, bridges and dams is very important. It is so necessary to provide for the evolution of the water flow depth and study the interaction between flow, transport dynamics and how those relationship influences changes. The changes in water flow sometimes affects crop fields, cause flooding, erosion and other damages like erosion, structures stability, so, to know effects of the water depth evolution process is very important [1]. The study of water depth evolution focuses on understanding the relationship that exists between the movement of water and the movement of sedimentary materials transported. It is therefore appropriate to develop an approach capable of follow water depth evolution in order to take the necessary steps. In the numerical simulation of water flow evolution problems, the mathematical model includes a hydrodynamical component coupled with a morphodynamical component [2, 4]. The Saint-Venant equations are used to predict the hydrodynamic behavior of water flows while the Exner equation is used to model sediment transport [3, 5]. Numerical simulation Saint Venant- Exner system involves different physical mechanisms, hence, having robust numerical schemes for the numerical simulation of this type of problem is need to accurately resolve both hydrodynamic and morphodynamic problem. However, most morphodynamical flows involve important features like moving fronts, stiff fronts, shock waves, discontinuities which are significant challenge to the accuracy and stability of numerical models. The principal aim is to find a stable, reliable, and accurate numerical method able to approximate solution of water depth evolution in time and space. But, in the zones where spatial activity moves in time such as, stiff moving fronts, shocks, we need to have a very fine grid to resolve those problems. In such cases, moving grids method can be a technique to improve efficiency and accuracy of numerical solutions. This work is devoted on numerical simulation water depth evolution phenomenon by the moving grid method under the method of lines.

The paper is organized as follow: **Section 2** is consecrated to the review the Saint Venant-Exner equation system, **Section 3** is a recall of moving grids method and the numerical schemes for the simulation, in **Section 4**, we present numerical results for three test problems to assess both the efficiency and accuracy of the schemes. The paper ends with concluding remarks in **Section 5**.

Governing Saint-Venant-Exner system

Description of Saint-Venant-Exner system: The Saint Venant equation is one of the hyperbolic systems of the conservation law that is used to solve water flow problems. The water flow is generally accompanied by sediment transport phenomenon. To model this system, one considers a coupled model constituted by a hydrodynamical component and a morphodynamical component [6, 7]. The hydrodynamic model is described by the Saint-Venant equation and the morphodynamic model is described by the Exner equation which includes a conservation law related to the evolution of the bottom topography due to the fluid action [8, 9]. The governing equations are obtained under Saint-Venant system conditions and includes equations for the conservation of water mass and momentum of the water phase.

Hydrodynamic Component: For an inviscid and incompressible flow, Saint-Venant system can be expressed as follow [10]:

$$\begin{cases} \dfrac{\partial h}{\partial t}+\dfrac{\partial}{\partial x}(hu)=0, & (1)\\ \dfrac{\partial}{\partial t}(hu)+\dfrac{\partial}{\partial x}\left(hu^2+\dfrac{1}{2}gh^2\right)=gh(S-\tau), & (2)\end{cases}$$

where τ is bottom frictional terms.

Morphodynamic Component: The equation that describes sediment transport phenomenon is a continuity equation. In the bed load transport, mass conservation law called the Exner equation is used to follow the bed elevation and the equation is:

$$\frac{\partial B}{\partial t}+\xi\frac{\partial Q_s}{\partial x}=0 \qquad (3)$$

Governing system: For a fixed bed, by neglecting viscous and Coriolis effects, The Saint Venant-Exner system in one dimension is:

$$\begin{cases} \dfrac{\partial h}{\partial t}+\dfrac{\partial}{\partial x}(hu)=0\\ \dfrac{\partial}{\partial t}(hu)+\dfrac{\partial}{\partial x}\left(hu^2+\dfrac{1}{2}gh^2\right)=ghS\,,(4)\\ \dfrac{\partial B}{\partial t}+\rho\dfrac{\partial Q_s}{\partial x}=0\end{cases}$$

where$h$ is the water depth, u is the velocity component, g is the acceleration due to gravity, B represent the thicknessof sediment layer, $S=-\frac{\partial B}{\partial x}$ is the bed slope, discharge Q_s is the volumetricbedload sediment transport rate per unit time and width $\rho=\frac{1}{1-\gamma}$, where $\gamma$is the porosity of the sediment layer, the conservative variable hu is also called water discharge and noted by q[10, 11,12]. Many different formulae for predicting and estimating bedload transport rate have been developed and one of them is proposed by Grass [13]. Expression proposed by Grass considers $Q_s$as a function of the flow velocity and a coefficient which depends on soil properties. The expression is $Q_s=A_g u|u|^{m_g-1}$ where$1\leq m_g\leq 4$ and $0\leq A_g\leq 1$. The parameter A_g is the coefficient to control the interaction between the bed and the water flow [14, 15].

Moving grids methodand numerical schemes

Brief recall of moving grids method: Numerical techniques to solve PDEs evolving in time are most often based on a discretization of the spatial domain. The resulting mesh is generally fixed in time, but the needs of a given application may require the mesh itself to change as the system evolves, adapting to the physics problem [16]. The adaptive method is more efficient for numerical solution of partial differential equations (PDEs) that produces specific structures, such as stiff fronts, shock waves, or overflow, that are localized in space [17]. Adapting the mesh can prove computationally efficient in that an adaptive mesh generally requires fewer points than a fixed mesh to attain the same level of accuracy. The main idea of moving grids method is to relocate grid points in a mesh having a fixed number of nodes in such a way that the nodes remain concentrated in regions of rapid variation of the solution. The fundamental principle of moving grids method is the equidistribution principle proposed by de Boor, principle which offer an excellent error estimation principle when formulating moving grids equations. The grid points are moved so that a specified quantity, also called the monitor function isequally distributed over the spatialdomain [18]. In the moving grids method, the monitor function connecting the mesh with the physical solution, is chosen to redistribute more grid points at critical regions where more accuracy is needed there by reducing errors introduced by the numerical scheme [19, 20, 21]. In this paper, we use arc-length monitor function for numerical simulation and for more details on moving grids method, see [22].

Numerical formulation of the moving grid method: We now utilize the moving grid technique to determine numerical schemes which used for solving Saint Venant-Exner system. Suppose that $[a;\,b]$ is the physical domain with a physical variable x and $[0;\,1]$ is the computational domain for a computational variable ξ. The coordinates transform is expressed as follow:

$$x\;=\;x(\xi;\;t)\colon[0,1]\longrightarrow[a,b],t>0,x\in[a;b],\xi\in[0;1]$$

Thus, the solution $h,u,B$are transformed as:

$$\begin{aligned}&h(x;\;t)=h(x(\xi,t),t)\\&u(x;\;t)=u(x(\xi,t),t)\quad(5)\\&B(x;\;t)=B(x(\xi,t),t)\end{aligned}$$

The coordinate x is rearranged as follows:

$$x_i(\xi)=x(\xi_i,t),i=1,\dots,n+1.$$
$$\xi_i=\frac{(i-1)(b-a)}{n}\,,i=1,\dots,n+1.$$

The uniform mesh on $[0,1]$ is ξ_i and $a=x_1<x_2<\cdots<x_n<x_{n+1}=b$ is the corresponding mesh on physical domain. Applying the chain rule of the method

$$h_x=\frac{h_\xi}{x_\xi},\quad h_t=\dot{h}-\frac{h_\xi}{x_\xi}x_t$$
$$q_x=\frac{q_\xi}{x_\xi},\quad q_t=\dot{q}-\frac{q_\xi}{x_\xi}x_t$$
$$B_x=\frac{B_\xi}{x_\xi},\quad B_t=\dot{B}-\frac{B_\xi}{x_\xi}x_t$$

and posing that $q=hu$, $Q_s=u^3$, the Saint Venant-Exner system (4) can be written as follows:

$$\begin{cases}\dfrac{\partial h}{\partial t}-\dfrac{h_\xi}{x_\xi}x_t+\dfrac{q_\xi}{x_\xi}=0,(6)\\ \dfrac{\partial q}{\partial t}-\dfrac{q_\xi}{x_\xi}x_t+\dfrac{2hqq_\xi-q^2h_\xi}{h^2x_\xi}+gh\left(\dfrac{h_\xi}{x_\xi}+\dfrac{B_\xi}{x_\xi}\right)=0,(7)\\ \dfrac{\partial B}{\partial t}-\dfrac{B_\xi}{x_\xi}x_t+\dfrac{3A_g}{1-\gamma}\left(\dfrac{q}{h}\right)^2\dfrac{hq_\xi-qh_\xi}{h^2x_\xi}=0, \qquad (8)\end{cases}$$

Employed method of lines approach and using central finite difference scheme for space variable discretization, the system of ODEs obtained is as follows:

$$\begin{cases}\dfrac{dh_i}{dt}-\dfrac{h_{i+1}-h_{i-1}}{x_{i+1}-x_{i-1}}\dfrac{dx_i}{dt}+\dfrac{q_{i+1}-q_{i-1}}{x_{i+1}-x_{i-1}}=0; \qquad i=2,\dots,n\\ \dfrac{dq_i}{dt}-\dfrac{q_{i+1}-q_{i-1}}{x_{i+1}-x_{i-1}}\dfrac{dx_i}{dt}+\dfrac{2q_i}{h_i}\dfrac{q_{i+1}-q_{i-1}}{x_{i+1}-x_{i-1}}-\left(\dfrac{q_i}{h_i}\right)^2\dfrac{h_{i+1}-h_{i-1}}{x_{i+1}-x_{i-1}}+gh_i\left(\dfrac{h_{i+1}-h_{i-1}}{x_{i+1}-x_{i-1}}-\dfrac{B_{i+1}-B_{i-1}}{x_{i+1}-x_{i-1}}\right)=0,(9)\\ \dfrac{dB_i}{dt}-\dfrac{B_{i+1}-B_{i-1}}{x_{i+1}-x_{i-1}}\dfrac{dx_i}{dt}+\dfrac{3A_g}{1-\gamma}\left(\dfrac{q_i}{h_i}\right)^2\left[\dfrac{1}{h_i}\dfrac{q_{i+1}-q_{i-1}}{x_{i+1}-x_{i-1}}-\dfrac{q_i}{h_i^{\,2}}\dfrac{h_{i+1}-h_{i-1}}{x_{i+1}-x_{i-1}}\right]=0\end{cases}$$

We apply MATLAB-based Method of Lines (MATMOL) toolbox specially MATLAB solver ode15s and for the study case, we give summary computational statistics using the following notations:

$\boldsymbol{n}$: moving grid node number,
$\boldsymbol{nr}$: grid fixe node number,
$\boldsymbol{STEPS}$: number of successful steps,
$\boldsymbol{FAIL}$: number of failed attempts,
$\boldsymbol{FNS}$: number of function evaluations,
$\boldsymbol{PDR}$: number of partial derivatives,
$\boldsymbol{LU}$: number of LU decompositions,
$\boldsymbol{LIN}$: number of solutions of linear system,
$\boldsymbol{CPU}$: CPU-time.

NUMERICAL RESULTS

In this section we present numerical results obtained for two examples. The exact solution is unknown, so we compare the computed solutions with a reference one obtained by using a very fine meshfor both h, q and B.

The constants values are:

$\gamma = 0.4;\ A_g = 0.3;\ g = 9.812;\ a = -10;\ b = 10.$

Like the exact solution is unknown, we compare the numerical solutions with $n = 300$ to the reference one obtained by using a very fine mesh $nr = 2150$ cells.

The following figures shows water depth evolution at $t = 0.3;\ 1\ and\ 1.5$

In Figure 1, it can be seen that, the numericalresults are in good agreement with corresponding reference solution and Table 1 show also that the results are satisfactory with this example.

Example 2

The example is a case of a transcritical flow with a shock over a parabolic bump [25, 26]. The initial conditions are given by:

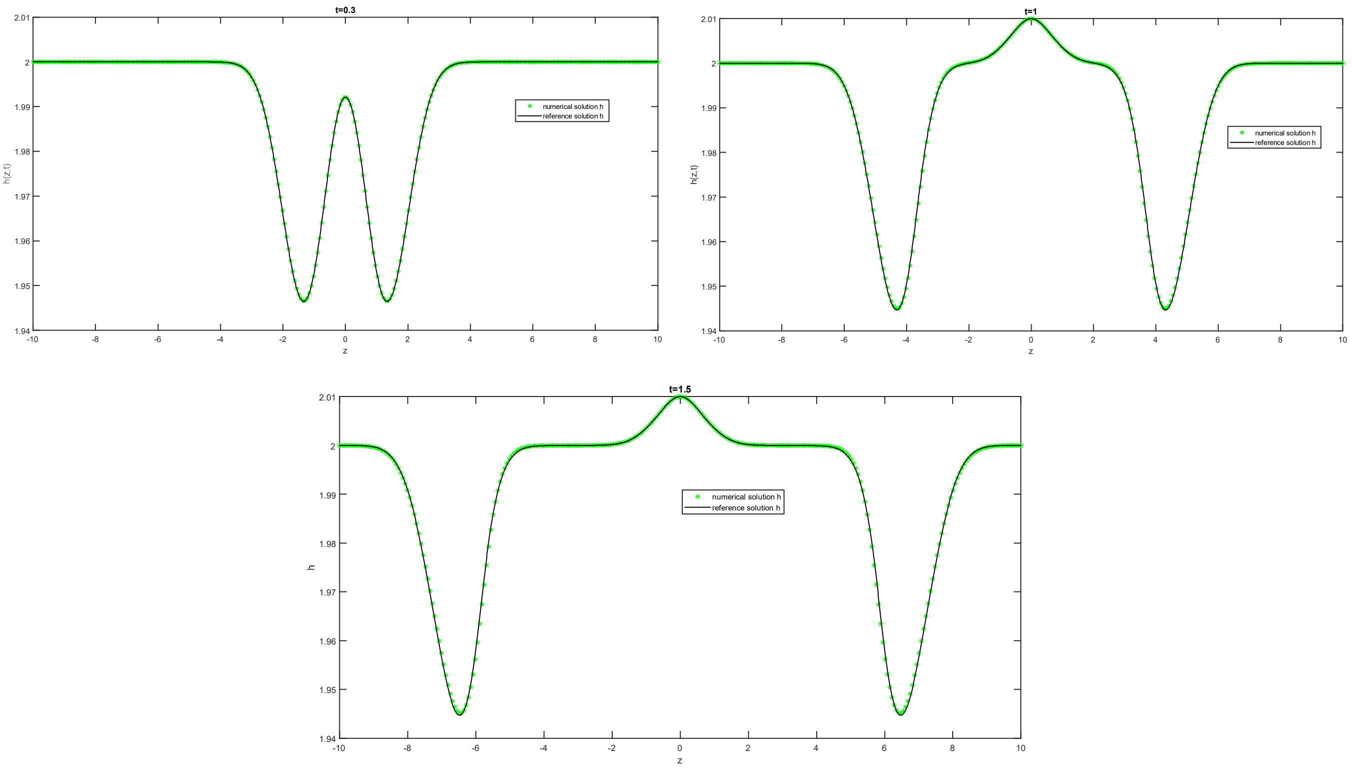

Figure 1. Comparison of solutions obtained with a moving grid for $n = 300$ nodes and those of a uniform fixed grid for$nr = 2150$ nodes

Table 1. Saint Venant-Exner couple system with strong interaction: numerical statistics, moving grid with $n = 300$ nodes, uniform fixed grid with $nr = 2150$ nodes, respectively.

	$Suc.St$	$Fail.at$	$Fun.ev$	$Part.der$	$LU.dec$	$Sol.lin$	$CPU.t$
$n = 300$	115	5	188	1	20	1168	13.3260
$nr = 2150$	105	6	146	1	12	124	95.5842

Example 1: We considera strong interaction between sediment layer and water flow in one-dimensional channel with flat bed along the interval $[-10, 10]$ under the following initialconditions [7, 10, 24]:

$h(x,0) = 2 - 0.1e^{-x^2}$

$q(x,0) = 0$

$B(x,0) = 0.1 + 0.1e^{-x^2}$

The boundary conditions are: $q(a,t) = 0; h(b,t) = 0$

$$h(x,0) = \begin{cases} 0.13 + 0.05(x-10)^2, & if\ 8 < x < 12 \\ 0.33 & othewise \end{cases}$$

$q(x,0) = 0.18$

$$B(x,0) = \begin{cases} 3 - 0.05(x-10)^2, & if\ 8 < x < 12 \\ 2.8 & otherwise \end{cases}$$

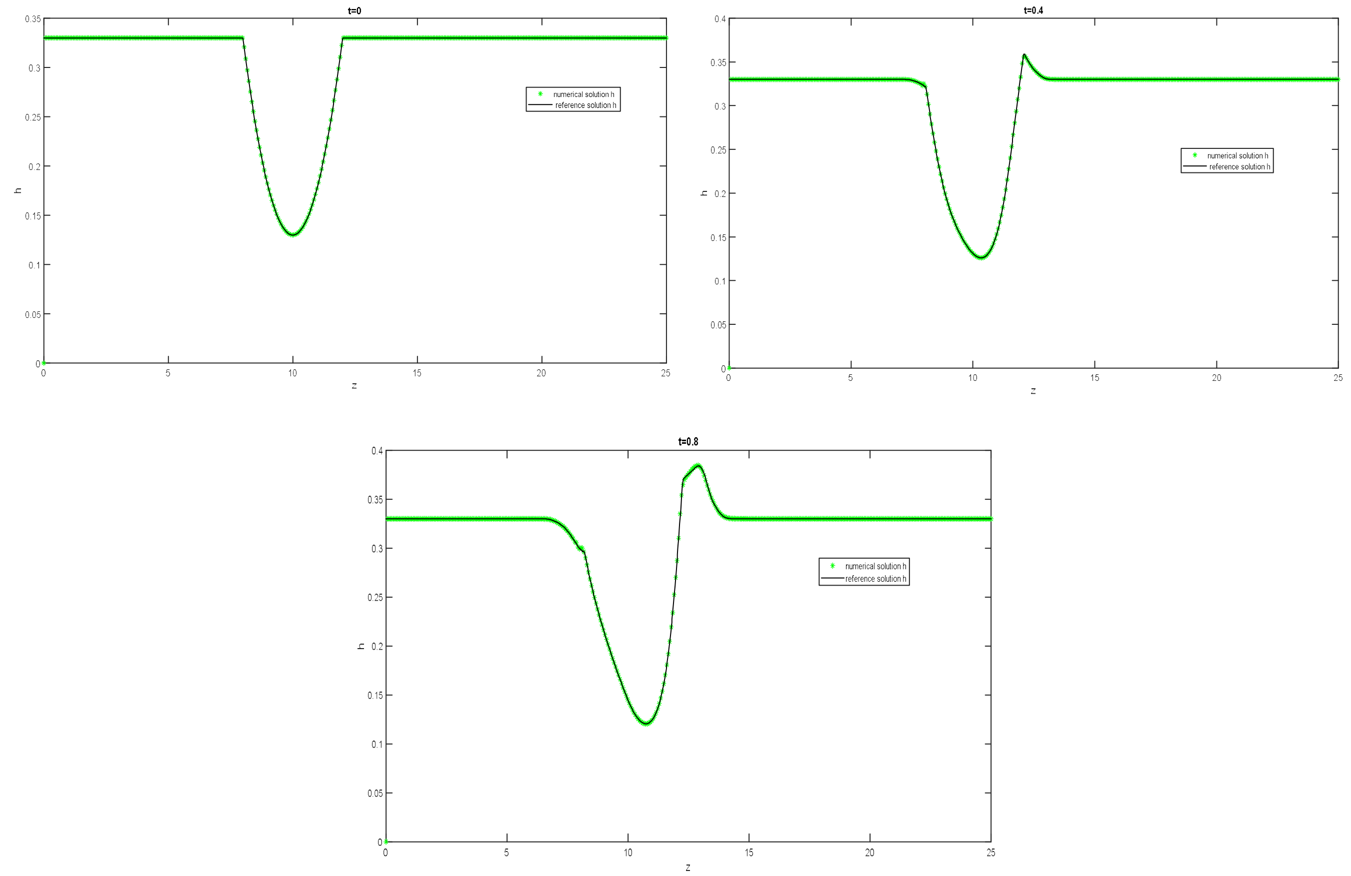

Figure 2: Comparison of solutions obtained with a moving grid for $n = 400$ nodes and those of a uniform fixed grid for $nr = 2000$ nodes

Table 2. Computational statistics for the transcritical flow with a shock case

	$Suc.St$	$Fail.at$	$Fun.ev$	$Part.der$	$LU.dec$	$Sol.lin$	$CPU.t$
$n = 400$	46	1	98	1	14	78	11.8438
$nr = 2000$	34	2	98	2	10	63	82.5315

The boundary conditions are given by:

$q(a,t) = 0.18;\ h(b,t) = 0.33$

The constants values are:

$\gamma = 0.4;\ A_g = 0.1;\ g = 9.812;\ a = 0;\ b = 25.$

We compute the numerical solution using $n = 400$ points in the interval $[0, 25]$ and compare the results with the reference solution computed on a fine grid with $nr = 2000$ points for small time. The water depth evolution at different times$t = 0;\ 0.4\ and\ 0.8$ are shown in Figure 2. In figure 2, we remark that when the mesh is very refined the numerical solutions converge to the reference solution. Table 2 shows that the method gives satisfactory results.

CONCLUSION

In this paper, we have discussed on moving grids technique for numerical approximation solution of the coupled system Saint Venant-Exner equations that govern water flow depth and interactions with the other components of this system. We simulated numerically the coupled system Saint Venant-Exner with moving grids method by using two test problems. Since the exact solution is unknown, the computed solutions are compared with a reference one obtained using a very fine mesh. From the test problems, satisfactorynumerical accuracy and efficiency properties are observed. The results of the proposed scheme based on moving grid techniques in this paper shown that, numerical solutions obtained are in a good agreement with the reference solutions.

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